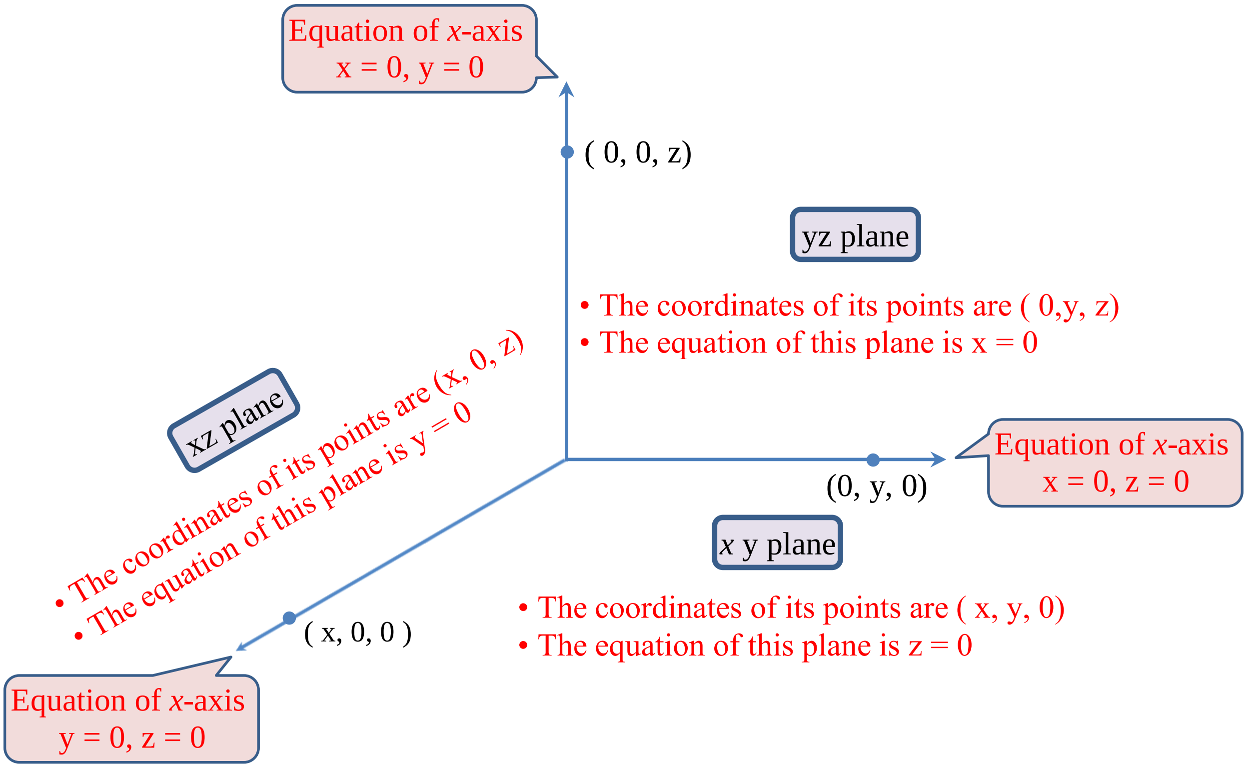
Three-dimensional orthogonal coordinate system



A (x, y, z)

The projection of A on:
- x y plane is (x, y, 0)
- yz plane is (0, y, z)
- x z plane is (x, 0, z)

The projection of A on:
- x-axis is (x, 0, 0)
- y-axis is (0, y, 0)
- z-axis is (0, 0, z)

A (x, y, z)

Distance between A and:
- x y plane $= \sqrt{z^2} = |z|$
- yz plane $= \sqrt{x^2} = |x|$
- x z plane $= \sqrt{y^2} = |y|$

Distance between A and:
- x-axis =
- y-axis $= \sqrt{x^2 + z^2}$
- z-axis $= \sqrt{x^2 + y^2}$

If A (x_1, y_1, z_1) and B (x_2, y_2, z_2) are two points in space, then

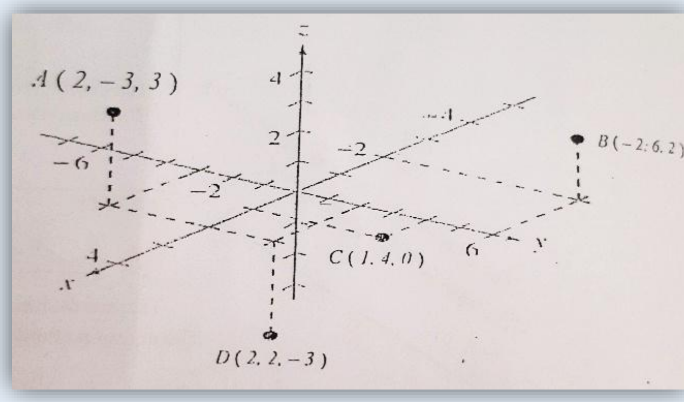
The distance between two points is $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$

The coordinates of the mid-point of $\overline{AB} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$

EXAMPLE! (1)

Plot each point in space
i) A (2, -3 , 3) ii) B (- 2 , 6 , 2) iii) C (1, 4 , 0) iv) D (2 , 2 , -3)

SOLUTION



EXAMPLE! (2)

Find the distance between the point (- 2 , 3 , 4) and
(a) xy plane (b) yz plane
(c) xz plane (d) x-axis
(e) y-axis (f) z- axis

SOLUTION

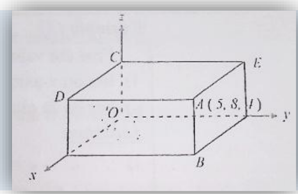
(a) the distance between the point (-2 , 3 , 4) and xy plane = $|z| = |4| = 4$
(b) the distance between the point (-2 , 3 , 4) and yz plane = $|x| = |-2| = 2$
(c) the distance between the point (-2 , 3 , 4) and xz plane = $|y| = |3| = 3$
(d) the distance between the point (-2 , 3 , 4) and x-axis = $\sqrt{(3)^2 + (4)^2} = 5$
(e) the distance between the point (-2 , 3 , 4) and y-axis = $\sqrt{(-2)^2 + (4)^2} = 2\sqrt{5}$
(f) the distance between the point (-2 , 3 , 4) and z-axis = $\sqrt{(-2)^2 + (3)^2} = \sqrt{13}$

EXAMPLE! (3)

The opposite figure represents a cuboid in an orthogonal system. two of the vertices of the cuboid are O (0, 0, 0) and A (5, 8, 4), find the coordinates of each of the following points B, C, D, E.

SOLUTION

B (5, 8, 0)
C (0, 0, 4)
D (5, 0, 4)
E (0, 8, 4)



EXAMPLE! (4)

Complete

(a) the two straight lines $\overleftrightarrow{xx'}$ and $\overleftrightarrow{yy'}$ determine the … plane whose equation is …

(b) the two straight lines $\overleftrightarrow{yy'}$ and $\overleftrightarrow{zz'}$ determine the coordinate plane …. whose equation is ….

(c) the x z and y z planes intersect in ….

(d) the coordinate planes x y and y z intersect in …

(e) if the point (x, y, z) ∈ y z plane , then x = …

(f) the point (0, 5, -3) lies in … plane whose equation is ….

(g) the point (0, 3, 0) lies on … axis and the point (0, 0, 4) lies on … axis

(h) the point (2, 0, -3) lies in … plane and the point (1, -3, 0) lies in … plane

(i) the distance between the point (-3, 2, 5) and y z plane equals …. Units of length

(j) the length of the perpendicular segment drawn from the point (8, 6, -4) to z-axis = …. Unit of length

(k) the projection of the point (-1, 2, 3) on x-axis is the point …. And its projection on y z plane is the point ….

SOLUTION

(a) x y plane , z = 0

(b) y z plane , x = 0

(c) z-axis

(d) y-axis

(e) x = 0

(f) y z plane, x = 0

(g) y-axis, z-axis

(h) x z plane, x y plane

(i) 3

(j) 10

(k) (- 1, 0, 0), (0, 2, 3)

EXAMPLE! (5)

Find the value (s) of a if the point $(x, a^2 - 4, a - 2)$

i) lies on x-axis

ii) lies on x z plane

SOLUTION

i) ∵ $a^2 - 4 = 0$ and $a - 2 = 0$

∴ $a^2 = 4$ $a = 2$

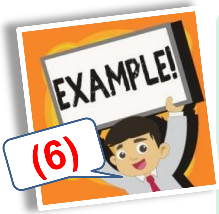
∴ $a = \pm 2$

∴ $a = 2$

ii) ∵ $a^2 - 4 = 0$

∴ $a^2 = 4$

∴ $a = \pm 2$



Find the coordinates of the point

(a) A (1 – k, 2k, 3+ k) where A ∊ xy plane

(b) B (2 + m, 3 - m, m) where B ∊ xz plane

(c) C (0, n + 5, n – 4) where C ∊ z -axis

(d) D (3 ℓ, ℓ + 3, 5 ℓ) where D ∊ y-axis

(e) E (ℓ + 5, 2 ℓ,) which is far from x-axis by 2 $\sqrt{5}$ units

(f) F (2 + k, 4k, - k) if its distance from yz plane is 5 units

SOLUTION

(a) A (1 – k, 2k, 3+ k) ∈ xy plane ⇒ z = 0 ⇒ 3 + k = 0 ⇒ k = - 3

∴ A = (4, - 6,0)

(b) B (2 + m, 3 - m, m) ∈ xz plane ⇒ y = 0 ⇒ 3 - m = 0 ⇒ m = 3

∴ B = (5, 0, 3)

(c) C (0, n + 5, n – 4) ∈ z-axis ⇒ x = 0 and y = 0 ⇒ n + 5 = 0 ⇒ n = - 5

∴ C = (0, 0, -9)

(d) D (3 ℓ, ℓ + 3, 5 ℓ) ∈ y-axis ⇒ x = 0 and z = 0 ⇒ 3 ℓ = 5 ℓ = 0 ⇒ ℓ = 0

∴ D = (0, 3, 0)

(e) E (ℓ + 5, 2 ℓ,) is far from x-axis by 2 $\sqrt{5}$ ⇒ $\sqrt{(2\ell)^2 + \ell^2} = 2\sqrt{5}$

⇒ $|\ell|\sqrt{5} = 2\sqrt{5}$

⇒ $\ell = \pm 2$

∴ E = (7, 4, 2) or E = (3, -4, -2)

(f) ∵ F (2 + k, 4k, - k) is far from yz plane by 5 units

∴ $|x| = 5$ ⇒ $|2 + k| = 5$

2 + k = 5 or 2 + k = -5

∴ k = 3 k = - 7

∴ F (5, 12, - 3) F (- 5, - 28, 7)

Note

Consider the three points A B C and AB, BC and AC are the lengths of the segments $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$ respectively.

* if the sum of the lengths of two segments = the length of the third segment, then A, B, C are collinear.

* if the sum of the lengths of two segments > the length of the third segment, then ABC is a triangle, and if AC (say) is the length of the greatest side, then

$(AC)^2 = (AB)^2 + (BC)^2$ ⇒ $\hat{B}$ is right.

$(AC)^2 > (AB)^2 + (BC)^2$ ⇒ $\hat{B}$ is obtuse.

$(AC)^2 < (AB)^2 + (BC)^2$ ⇒ $\hat{B}$ is acute.

EXAMPLE! (7)

S.B. 2017

Prove that the points A (4, 4, 0), B (4, 0, 4) and C (0, 4, 4) are the vertices of an equilateral triangle and find its area.

SOLUTION

$AB = \sqrt{(4-4)^2 + (4-0)^2 + (0-4)^2} = 4\sqrt{2}$

$BC = \sqrt{(4-0)^2 + (0-4)^2 + (4-4)^2} = 4\sqrt{2}$

$AC = \sqrt{(4-0)^2 + (4-4)^2 + (0-4)^2} = 4\sqrt{2}$

$\because AB = BC = AC$

$\therefore \triangle ABC$ is an equilateral triangle

Area of $\triangle ABC = \frac{1}{2}(AB)(BC)\sin 60$

$= \frac{1}{2} \times 4\sqrt{2} \times 4\sqrt{2} \times \frac{\sqrt{3}}{2}$

$= 8\sqrt{3}$ units of area

EXAMPLE! (8)

S.B. 2017

Prove that the triangle ABC where A (2, - 1 , 3) , B (- 4 , 4, 2) and C (-2, 5, 1) is right angled at C.

SOLUTION

$(AB)^2 = (2+4)^2 + (-1-4)^2 + (3-2)^2 = 62$

$(BC)^2 = (-4+2)^2 + (4-5)^2 + (2-1)^2 = 6$

$(AC)^2 = (2+2)^2 + (-1-5)^2 + (3-1)^2 = 56$

$\because (AB)^2 = 62$ and $(BC)^2 + (AC)^2 = 6 + 56 = 62$

That is $(AB)^2 = (BC)^2 + (AC)^2$

$\therefore m(\hat{C}) = 90$

EXAMPLE! (9)

S.B. 2017

Find the point on y-axis equidistant from the points A (2, 5, - 3), B (- 3, 6, 1)

SOLUTION

Let the point A (2, 5, - 3), B (- 3, 6, 1) and a point C (0, y, 0) on the y-axis

C (0, y, 0) be a point on y-axis

$\because CA = CB$

$\therefore \sqrt{(0-2)^2 + (y-5)^2 + (0+3)^2} = \sqrt{(0+3)^2 + (y-6)^2 + (0-1)^2}$

$\therefore \sqrt{13 + (y-5)^2} = \sqrt{10 + (y-6)^2}$

$\therefore 13 + (y-5)^2 = 10 + (y-6)^2$

$\therefore 13 + y^2 - 10y + 25 = 10 + y^2 - 12y + 36$

$\therefore 38 - 10y = 46 - 12y$

$\therefore 2y = 8$ $\qquad \therefore y = 4$

$\therefore$ the required point is C (0, 4, 0)

EXAMPLE! (10)

S.B. 2017

If A (1, -3, 2) and B (4, - 1, 4), find the coordinates of the mid-point of $\overline{AB}$

SOLUTION

the coordinates of the mid-point = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

$= \left(\frac{1+4}{2}, \frac{-3-1}{2}, \frac{2+4}{2}\right)$

$= \left(\frac{5}{2}, -2, 3\right)$

EXAMPLE! (11)

If C (2, 2, 6) is the mid-point of $\overline{AB}$ where A (1, - 4, 0) , find the coordinates of the point B

SOLUTION

Let the point B (x, y, z)

∵ C is the mid-point of $\overline{AB}$

∴ (2, 2, 6) = $\left(\frac{x+1}{2}, \frac{y-4}{2}, \frac{z+0}{2}\right)$

∴ $\frac{x+1}{2} = 2$ | $\frac{y-4}{2} = 2$ | $\frac{z+0}{2} = 6$

∴ x = 3 | y = 8 | z = 12

∴ B (3, 8, 12)

EXAMPLE! (12)

If C (- 1, 6, - 5) is the mid-point of $\overline{AB}$ where A (k - 2, - 1, m+ 3) , of the point B (2, n – 7 , - 2) , find the value of k + n – m

SOLUTION

∵ C is a mid-point of $\overline{AB}$

∴ (- 1, 6, -5) = $\left(\frac{k-2+2}{2}, \frac{n-7-1}{2}, \frac{m+3-2}{2}\right)$

∴ $\frac{k}{2} = -1$ | n – 8 = 12 | m +1 = -10

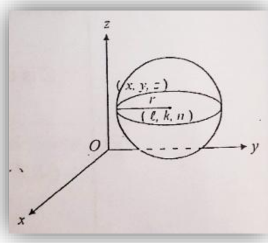
∴ k = -2 | n = 20 | m = - 11

∴ k + n – m = - 2 + 20 + 11 = 29

The sphere

The sphere is defined as the set of point in space which are equidistant from fixed point (the center of the sphere), the distance between the center and any point of the sphere is called the radius of the sphere.

The standard from of the equation of a sphere



If the point (x, y, z) lies on the sphere of center (ℓ, k, n) and radius r, then r = $\sqrt{(x - \ell)^2 + (y - k)^2 + (z - n)^2}$

squaring both sides, we get

$(x - \ell)^2 + (y - k)^2 + (z - n)^2 = r^2$

The general equation of the sphere

$x^2 + y^2 + z^2 + 2\ell x + 2ky + 2nz + d = 0$ where = $\ell^2 + k^2 + n^2 - r^2$

This equation represents a sphere whose center is $\left(\frac{coeff.of\ x}{-2}, \frac{coeff.of\ y}{-2}, \frac{coeff.of\ z}{-2}\right)$ and radius r = $\sqrt{\ell^2 + k^2 + n^2 - d}$ where $\ell^2 + k^2 + n^2 > d$

Remark

(1) If the spere of center (ℓ, k, n) touches $\begin{cases} xy\ palnc, then\ r = |n| \\ xz\ palnc, then\ r = |k| \\ zy\ palnc, then\ r = |\ell| \end{cases}$

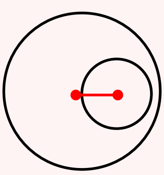
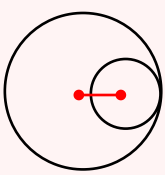
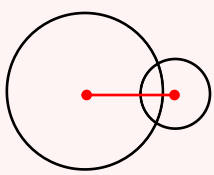
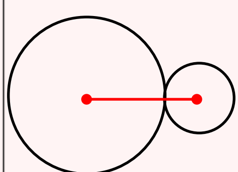
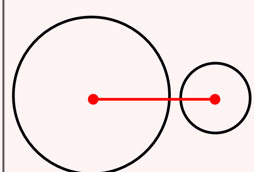
(2) if the sphere touches the cartesian planes and the coordinates of its center are all positive numbers, then the coordinates of its center are (r, r, r)

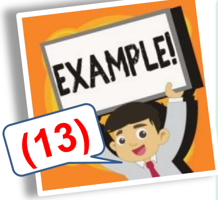
(3) if the sphere touches the cartesian axes and the coordinates of its center are all positive numbers, then the coordinates of its center are $\left(\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right)$

(4) the cross section of a sphere with a plane is a circle.

Remark

M_1 , M_2 are the centers of two spheres and are r_1 , r_2 are their radii where $r_1 > r_2$

$M_1 M_2 < r_1 - r_2$	$M_1 M_2 = r_1 - r_2$	$r_1 - r_2 < M_1 M_2 < r_1 + r_2$	$M_1 M_2 = r_1 + r_2$	$M_1 M_2 > r_1 + r_2$
				
M_2 inside M_1	Touch each other internally	Intersecting sphere	Touch each other externally	Distant sphere



S.B. 2017

Find the standard from of the equation of the sphere

(a) whose center is (2, - 1, 4) and its radius is 3 units .

(b) whose center is the origin and its radius is 5 units.

(c) in which A (-1, 5, 4) and B (5, 1 -2) are the end point of a diameter.

(a) the standard form of the equation of the sphere is

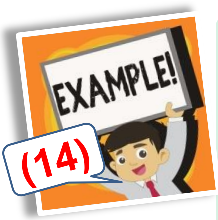
$(x - 2)^2 + (y + 1)^2 + (z - 4)^2 = 9$

(b) the equation of the sphere is $x^2 + y^2 + z^2 = 25$

(c) the center of the sphere is the mid-point of $\overline{AB} = \left(\frac{-1+5}{2}, \frac{5+1}{2}, \frac{4-2}{2}\right) = (2, 3, 1)$

$\therefore r = \sqrt{(2+1)^2 + (3-5)^2 + (1-4)^2} = \sqrt{22}$

$\therefore$ the equation of the sphere is $(x - 2)^2 + (y - 3)^2 + (z - 1)^2 = 22$



S.B. 2017

Find the center and the radius of each of the following spheres

(a) $x^2 + y^2 + z^2 + 4x - 2y - 6z + 11 = 0$

(b) $x^2 + y^2 + z^2 = 6z$

(c) $x^2 + y^2 + z^2 - 2x + 4y = 0$

(d) $3x^2 + 3y^2 + 3z^2 - 3x - 9y - 6z + 3 = 0$

(a) the center is (-2, 1, 3)

$\therefore r = \sqrt{(-2)^2 + (1)^2 + (3)^2 - 11} = \sqrt{3}$

(b) the center is (0, 0, 3)

$\therefore r = \sqrt{0 + 0 + 9 - 0} = 3$

(c) the center is (1, -2, 0)

$\therefore r = \sqrt{(1)^2 + (-2)^2 + (0)^2 - 0} = \sqrt{5}$

d) $\because 3x^2 + 3y^2 + 3z^2 - 3x - 9y - 6z + 3 = 0$ $\div (3)$

$\therefore x^2 + y^2 + z^2 - x - 3y - 2z + 1 = 0$

$\therefore$ the center is $\left(\frac{1}{2}, \frac{3}{2}, 1\right)$

$\therefore r = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + (1)^2 - 1} = \sqrt{\frac{10}{4}} = \sqrt{\frac{10}{2}}$

EXAMPLE! (15)

Find the equation of the sphere whose center is
(a) (-5, 1, -2) and touches xy plane
(b) (0, 4, -2) and touches xz plane

SOLUTION

(a) $r = |-2| = 2$ units of length

$\therefore$ the equation of the sphere is $(x+5)^2 + (y-1)^2 + (z+2)^2 = 4$

(b) $r = |4| = 4$ units of length

$\therefore$ the equation of the sphere is $x^2 + (y-4)^2 + (z+2)^2 = 16$

EXAMPLE! (16)

Find the equation of the sphere whose radius is 2 units and touches the cartesian planes given that the coordinates of the center of the sphere are positive.

SOLUTION

The center is (2, 2, 2) and are $r = 2$

The equation of the sphere is

$(x-2)^2 + (y-2)^2 + (z-2)^2 = 4$

EXAMPLE! (17)

If the x-axis cuts the sphere $(x-2)^2 + (y+3)^2 + (z-1)^2 = 14$ at the two point A and B, find the length of $\overline{AB}$

SOLUTION

To find the intersection point of the x-axis and the sphere, put $y = 0$, $z = 0$

$\therefore (x-2)^2 + 9 + 1 = 14$

$\therefore (x-2)^2 = 4$

$\therefore x - 2 = 2$ or $x - 2 = -2$

$X = 4$ $\quad$ $x = 0$

$\therefore$ A (4, 0, 0) $\quad$ B = (0, 0, 0)

$\therefore$ AB = 4 units of length

EXAMPLE! (18)

The equation of two sphere are $(x-2)^2 + (y+4)^2 + (z-2)^2 = 1$ an $(x+4)^2 + (y-4)^2 + (z-2)^2 = 4$, find the distance between their centers and show that they are not intersecting

SOLUTION

$M_1 = (2, -4, 2)$, $\quad$ $M_2 = (-4, 4, 2)$

$r_1 = 1$, $\quad$ $r_2 = 2$

$r_1 + r_2 = 3$, $\quad$ $r_2 - r_1 = 1$

$M_1 M_2 = \sqrt{(2+4)^2 + (-4-4)^2 + (2-2)^2} = 10$ units

$\therefore M_1 M_2 > r_1 + r_2$

$\therefore$ the two sphere are not intersecting

EXAMPLE! (19)

If the two-sphere touching each other are $(x - 1)^2 + y^2 + (z - 3)^2 = 16$ and $(x + 1)^2 + (y - 2)^2 + (z - n)^2 = 25$, find the value of n .

SOLUTION

$M_1 = (1, 0, 3)$, $r_1 = 4$ and $M_2 = (-1, 2, n)$, $r_2 = 5$

$M_1 M_2 = \sqrt{(1+1)^2 + (0-2)^2 + (3-n)^2} = \sqrt{8 + (3-n)^2}$

$\because$ the two sphere touch each other

Externally **or** **Internally**

Externally:

$M_1 M_2 = r_1 + r_2$

$\sqrt{8 + (3-n)^2} = 9$

$\therefore 8 + (3 - n)^2 = 81$

$\therefore (3 - n)^2 = 73$

$\therefore 3 - n = \pm\sqrt{73}$

$\therefore n = 3 \pm \sqrt{73}$

Internally:

$M_1 M_2 = r_1 - r_2$

$\sqrt{8 + (3-n)^2} = 1$

$8 + (3 - n)^2 = 1$

$(3 - n)^2 = -7$

(x)

EXAMPLE! (20)

Find the equation of cross section (trace) of the sphere whose equation is $(x - 3)^2 + (y - 2)^2 + (z + 4)^2 = 25$ with xy-plane, find the area of the section

SOLUTION

$\because$ every point on xy plane is in the form (x, y, 0)

Put z = 0 is the equation of the sphere

$\therefore (x - 3)^2 + (y - 2)^2 + 16 = 25$

i.e. the equation of the crosse section is

$(x - 3)^2 + (y - 2)^2 = 9$

$\because r = \sqrt{9} = 3$

$\therefore$ the area of the section = Area of the circle

$= \pi r^2$

$= \pi (3)^2$

$= 9\pi$ squared units

EXERCISE 1

S.B.2017

1) find the distance between the two point A and B in each of the following :

(a) A (7, 0, 4), B (1, 0, 0) [$2\sqrt{13}$]

(c) A (1, 1, - 7), B (-2, -3, -7) [$2\sqrt{13}$]

S.B.2017

2) find the coordinates of the mid-point of $\overline{AB}$ in each of the following

(a) A (3, -1, 4), B (2, 0, -1) $\left[\left(\frac{5}{2}, \frac{-1}{2}, \frac{3}{2}\right)\right]$

(b) A (- 3, 5, 5), B (- 6, 4, 8) $\left[\left(\frac{-9}{2}, \frac{9}{2}, \frac{13}{2}\right)\right]$

S.B.2017

3) if C (- 1 , 4, 0) is the mid-point of $\overline{AB}$ where B (4, -2, 1), find the coordinates of point A [(- 6, 10, - 1)]

S.B.2017

4) find the equation of the sphere if

(a) the center is point (3, -1, 2), and its radius length is $\sqrt{7}$

[$(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = 2$]

(b) (3, 4, - 3), (0, 2, 1) are terminals of the diameter.

$\left[\left(x - \frac{3}{2}\right)^2 + (y - 3)^2 + (z + 1)^2 = \frac{29}{4}\right]$

(c) the center is point (1, -6, 1)and passe through point (2, -1, 5)

[$(x - 1)^2 + (y + 6)^2 + (z - 1)^2 = 42$]

S.B.2017

5) find the center and the radius length of the sphere in each of the following

(a) $x^2 + y^2 + z^2 = 9$ [(0, 0, 0), 3]

(c) $x^2 + y^2 + z^2 - 2x + 4y = 0$ [(1, - 2, 0), $\sqrt{5}$]

S.B.2017

6) find the equation of the sphere whose radius is 3 units and touches the cartesian planes (given that the three coordinates of the center are positive . [$(x - 3)^2 + (y - 3)^2 + (z - 3)^2 = 9$]

S.B.2017

7) if x-axis cuts the sphere $(x - 2)^2 + (y + 3)^2 + (z - 1)^2 = 14$ at the two points A and B find the length of $\overline{AB}$ [4]

S.B.2017

8) if all point in the space in the form of (x, y, 0) lie in xy plane whose equation is z = 0 , find the equation of the plane in which all of its point are in the form (x, y, 2) [z = 2]

Choose the following:

S.B.2017

9) the straight lines $\overleftrightarrow{xx}$ ‘ and $\overleftrightarrow{zz}$ ‘ form the coordinate plane whose equation is ……
(a) x = 0 (b) y = 0 (c) z = 0 (d) y = 2

S.B.2017

10) the distance between point (3, -1, 2) and the cartesian xz plane is …… unit length.
(a) 3 (b) -1 (c) 2 (d) 1

S.B.2017

11) xy plane ad yz plane intersect at …..
(a) origin point (b) x-axis (c) y-axis (d) z-axis

S.B.2017

12) the length of the perpendicular line drawn from point (-2, 3, 4) to x-axis is ……. Unit length.
(a) 2 (b) 3 (c) 5 (d) 4

S.B.2017

13) the coordinates of the mid-point of the line segment whose terminals are (-3, 2, 4), (5, 1, 8) are ……
(a) $\left(1, \frac{3}{2}, 6\right)$ (b) (2, -1, 4) (c) (8, -1, 4) (d) $\left(1, -\frac{3}{2}, 2\right)$

S.B.2017

14) the equation of the sphere whose center is the origin and the radius length is 5 units …..
(a) $x^2 + y^2 + z^2 = 5$
(b) $x^2 + y^2 + z^2 = 0$
(c) $(x - 5)^2 + (y - 5)^2 + (z - 5)^2 = 25$
(d) $x^2 + y^2 + z^2 = 25$

S.B.2017

15) the equation of sphere with center (2, -3, 4) and touches xy plane is
(a) $(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 4$
(b) $(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 9$
(c) $(x - 2)^2 + (y + 3)^2 + (z - 4)^2 = 16$
(d) $(x + 2)^2 + (y - 3)^2 + (z + 4)^2 = 16$

S.B.2017

16) if the point A (-1, -1, 2), find the coordinates of the point B which lies on y-axis , given that AB = $\sqrt{14}$ units of length .
[(0, 2, 0) or (0, -4 , 0)]

S.B.2017

17) ABC is triangle in which A (-4, -1, 2), B (3, 5, - 5) find the point C given that the mid-point of $\overline{AC}$ lies on y-axis and mid-point of $\overline{BC}$ lies in the plane xz
[(4, -5 , -2)]

S.B.2017 — 18) if the point (7, 1, 3), (5, 3, k), (3, 5, 3) are the vertices of triangle , prove that this triangle is isosceles , then find the value of k which make the triangle equilateral . [$3 \pm 2\sqrt{6}$]

S.B.2017 — 19) if A ∈ x-axis , B ∈ y-axis, C ∈ z-axis and if point (1, -1 , 0) is the mid-point of $\overline{AB}$ and point (0, -1, 2) is the mid-point of $\overline{BC}$, find the coordinates of the mid-point of $\overline{AC}$ [(1, 0 , 2)]

S.B.2017 — 20) if $(x - 2)^2 + (y + 4)^2 + (z - 2)^2 = 1$ and $(x + 4)^2 + (y - 4)^2 + (z - 2)^2 = 4$ are the equations of two sphere , find the distance between the centers of the two spheres and show that the two sphere do not intersect . [10]

S.B.2017 — 21) if the two spheres $(x - 3)^2 + y^2 + (z - 3)^2 = 16$ and $(x + 1)^2 + (y - 4)^2 + (z - k)^2 = 25$ are tangential , find the value of k [-4 or 20]

MOE 2017 — 22) if a sphere touches the planes xz, xy and yz at the points A,B and C respectively and $\overline{AD}$ is a diameter where D (3, 6, 3), find the equation of the sphere [$(x - 3)^2 + (y - 3)^2 + (z - 3)^2 = 9$]

S.B.2017 — 23) if A (7, -1, 8) and B (11, 2, - 4), then AB = ……… cm

(a) 10 (b) 11 (c) 12 (d) 13

S.B.2017 — 24) $x^2 + y^2 + z^2 + 4x - 6y + 8z + 4 = 0$ is an equation of a sphere whose diameter length = …… cm

(a) 5 (b) 10 (c) 15 (d) 20

S.B.2017 — 25) if A (7, -1, 8) and B (11, 2, - 4), then AB = ……… cm

(a) 10 (b) 11 (c) 12 (d) 13

S.B.2017 — 26) $x^2 + y^2 + z^2 + 4x - 6y + 8z + 4 = 0$ is an equation of a sphere whose diameter length = …… cm

(a) 5 (b) 10 (c) 15 (d) 20

S.B.2017 — 27) if $x^2 + y^2 + z^2 + 6x - 4y + 10z - 8 = 0$ is the equation of a sphere whose center is M, then M = ……

(a) (-3, 2, -5) (b) (4, -2, -5) (c) (-3, -2, -5) (d) (3, , 25)

S.B.2017 — 28) if (x, y, z) is the mid-point of AB , where A (-4, 0, 5) and B (-2, 4, -13), then x + y + z = ……..

(a) - 5 (b) -6 (c) 3 (d) 4

S.B.2017

29) the standard from of the equation of the sphere whose center is (3, -2, 1) and its radius length equals = 5 cm is ……..

(a) $(x + 3)^2 + (y - 2)^2 + (z + 1)^2 = 5$

(b) $(x + 3)^2 + (y - 2)^2 + (z + 1)^2 = 25$

(c) $(x – 3)^2 + (y + 2)^2 + (z – 1)^2 = 25$

(d) $(x - 3)^2 + (y + 2)^2 + (z - 1)^2 = \sqrt{5}$

MOE 2017

30) the radius length of the sphere $x^2 + y^2 + z^2 - 2x – 6y + 10z - 1 = 0$ equals ……. Units of length

(a) 3 (b) 4 (c) 5 (d) 6

MOE 2017

31) if A (2, -1 ,3) and B (-2, 2 ,- 9) , then the length of AB = ….. units of length

(a) 15 (b) 13 (c) 12 (d) 10

MOE 2017

32) if the y-axis cuts the sphere whose centre is (3, -4, 12) and its radius length equals 13 cm in the two point A and B, then AB equals …...

(a) 6 units of length (b) 8 units of length

(c) 24 units of length (d) 26 units of length

MOE 2017

33) the equation of the sphere whose center is M (1, -2 , -5) and is touched by the xy plane is ……

(a) $(x - 1)^2 + (y + 2)^2 + (z + 5)^2 = 1$

(b) $(x - 1)^2 + (y + 2)^2 + (z + 5)^2 = 4$

(c) $(x – 1)^2 + (y + 2)^2 + (z – 5)^2 = 25$

(d) $(x - 1)^2 + (y + 2)^2 + (z + 5)^2 = 25$

MOE 2017

34) A sphere whose centre is M is put inside a cube of an inner edge of length 10 cm such that the sphere touches all the faces of the cube . if one of the cube vertices is the origin point and three other vertices lie on the coordinate axes , then the equation of the sphere is ……

(a) $x^2 + y^2 + z^2 - 20x – 20y – 20z + 50 = 0$

(b) $x^2 + y^2 + z^2 + 10xy + 10yz + 50 = 0$

(c) $x^2 + y^2 + z^2 - 10x – 10y – 10z + 50 = 0$

(d) $x^2 + y^2 + z^2 + 10x – 10y – 10z + 50 = 0$

MOE 2017

35) the length of the diameter of sphere whose equation :

$x^2 + y^2 + z^2 - 6x + 8y – 4z + 4 = 0$ equals …….. units of length

(a) 5 (b) 10 (c) 15 (d) 20

MOE 2017	36) if x-axis cuts the sphere $(x - 2)^2 + (y + 3)^2 + (z - 1)^2 = 14$ in the two point A and B then the length of $\overline{AB}$ = ……. Units of length (a) 2 (b) $\sqrt{14}$ (c) 4 (d) $\sqrt{28}$
MOE 2017	37) the two straight lines $\overleftrightarrow{XX'}$, $\overleftrightarrow{YY'}$ form the cartesian plane whose equation is …… (a) $x = 0$ (b) $y = 0$ (c) $z = 0$ (d) $y = 2$
MOE 2017	38) the radius of the sphere $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$ equals …….. (a) 1 (b) 2 (c) 3 (d) $\sqrt{19}$

Complete the following:

S.B.2017	39) the equation of the sphere whose center is (2, - 3, 1) and its radius length equals $2\sqrt{5}$ is ……
S.B.2017	40) $x^2 + y^2 + z^2 - 4kx + 4y - 8z + 2k = 0$ is the equation of a sphere radius length $2\sqrt{5}$, then the value of k = ……..
S.B.2017	41) if C (-1, 9, -5) is the mid-point of AB, where A (k – 2, -1, m + 3) and B (2, n – 7, - 2), then k + m – n = …….
S.B.2017	42) the radius length of the sphere $x^2 + y^2 + z^2 - 4x - 6y + 8z + 4 = 0$ equals ……..
S.B.2017	43) the center of the sphere $x^2 + y^2 + z^2 + 8x - 12y + 2z + 1 = 0$ is ……..
S.B.2017	44) the length of the perpendicular drawn from the point (-2, - 3, 1)to the x-axis equals ……
S.B.2017	45) the standard form of the equation of the sphere whose center is (3, 4, - 5) and touches yz-plane is ……
S.B.2017	46) the radius length of the sphere $(x - 2)^2 + (y + 4)^2 + (z - 5)^2 = 64$ equals ……
S.B.2017	47) the radius length of the sphere: $x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0$ equals ……

Vector in space

The vector :

is a directed line segment determined by a magnitude and direction

* Position vector in space :

the position vector of the point A (A_x, A_y, A_z) with respect to the origin point 0(0,0,0) is the directed line segment whose starting point is the origin point and its end point is the point A

* the position vector of the point A is denoted by $\vec{A}$ = (A_x, A_y, A_z)

* A_x is the component of $\vec{A}$ in direction of x - axis

* A_y is the component of $\vec{A}$ in direction of y - axis

* A_z is the component of $\vec{A}$ in direction of z - axis

The norm of vector :

is the length of the directed line segment that represents this vector

*If $\vec{A}$ = (A_x, A_y, A_z) then $\|\vec{A}\| = \sqrt{(A_x)^2 + (A_y)^2 + (A_z)^2}$

(from rule of distance between two points)

If $\vec{A}$ = (2,-1, 3), $\vec{B}$ = (0,4, -3) then

SOLUTION

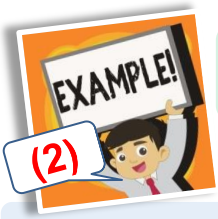
* component of $\vec{A}$ in direction of x - axis is 2

* component of $\vec{B}$ in direction of z - axis is - 3

$\|\vec{A}\| \, \| = \sqrt{4 + 1 + 9} = \sqrt{14}$ $\qquad$ $\|\vec{B}\| = \sqrt{0 + 16 + 9} = \sqrt{25} = 5$

* the vector $\vec{B}$ lies in the coordinate plane yz

(component of $\vec{B}$ in direction of x - axis vanishes).



If $\vec{A}$ = (-1, 4, 2), $\vec{B}$ = (3, 1,0) find : a) $A_x + B_y$ $\qquad$ b) $\|\vec{A}\| + \|\vec{B}\|$

SOLUTION

a) $A_x + B_y = -1 + 1 = 0$

b) $\|\vec{A}\| + \|\vec{B}\| = \sqrt{1 + 16 + 4} + \sqrt{9 + 1 + 0} = \sqrt{21} + \sqrt{10}$

Adding 3D vectors:

* if $\bar{A} = (A_x, A_y, A_z)$, $\bar{B} = (B_x, B_y, B_z,)$ then $\bar{C} = \bar{A} + \bar{B}$

$\bar{C} = (A_x + B_x, A_y + B_y, A_z, + B_z)$

$\bar{C} = (C_x, C_y, C_z)$

For example :

If $\bar{A} = (-2,3,1)$, $\bar{B} = (0, -2,4)$ then $\bar{A} + \bar{B} = (-2,1,5)$

If $\bar{A} = (4, -4,0)$, $\bar{B} = (-1,5,2)$ then $\bar{A} + \bar{B} = (3,1, 2)$

**** Properties of adding 3D vectors : For any two vectors A, B E R3 then:***

1) Closure property: $\bar{A} + \bar{B} \in R^3$
2) Commutative property: $\bar{A} + \bar{B} = \bar{B} + \bar{A}$
3) Associative property: $(\bar{A} + \bar{B}) + \bar{C} = \bar{A} + (\bar{B} + \bar{C})$
4) The neutral (identity) element over addition is the zero vector $\bar{0} = (0,0,0)$ in R^3 where $\bar{A} + \bar{0} = \bar{0} + \bar{A} = \bar{A}$
5) The additive inverse for any vector $\bar{A} = (A_x, A_y, A_z) \in R^3$ is $-\bar{A} = (-A_x, -A_y, -A_z) \in R^3$, where $\bar{A} + (-\bar{A}) = (-\bar{A}) + \bar{A} = \bar{0}$

** Multiplying a vector by a scalar (real number):*

If $\bar{A} = (A_x, A_y, A_z) \in R^3$ and $K \in R$, then:

$K\bar{A} = K(A_x, A_y, A_z) \in R^3 \rightarrow K\bar{A} = (KA_x, KA_y,, KA_z) \in R^3$

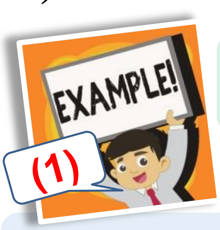
for example:

$3(2, -1,4) = (6 , -3 , 12)$, $\frac{1}{2}(4 , 9 , 6) = (2 , \frac{9}{2} , 3)$, $-2(1, -3 , -4) = (-2 , 6 , 8)$

Properties of multiplying a vector by a scalar (real number):

If $\bar{A}$ $\bar{B} \in R^3$ and k , $l \in R$ then

1) Distributive property: $k(\bar{A} + \bar{B}) = k\bar{A} + k\bar{B}$, $(k + l)\bar{A} = k\bar{A} + l\bar{A}$
2) Associative property: $k(l\bar{A}) = l(k\bar{A}) = (kl)\bar{A}$



If $\bar{A} = (-1, 5 , 2)$, $\bar{B} = (4 , -1, 3)$, find:

1) $2\bar{A} - 3\bar{B}$ 2) if $3\bar{A} + 2\bar{C} = 2\bar{B}$ find $\bar{C}$

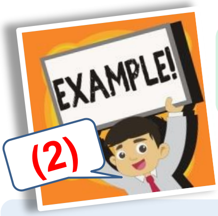
SOLUTION

1) $2\vec{A} - 3\vec{B} = 2(-1,5,2) - 3(4,-1, 3) = (-2, 10,4) - (12, -3,9) = (-14,13,-5)$

2) $3\vec{A} + 2\vec{C} = 2\vec{B}$

$2\vec{C} = 2\vec{B} - 3\vec{A} = 2(4,-1,3) - 3(-1,5,2) = (8, -2,6) - (-3,15,6)$

$= (11,-17,0) \therefore \vec{C} = \left(\frac{11}{2} , \frac{-17}{2} , 0\right)$



if $\vec{C}$ = (2, -3,1), $\vec{D}$ = (0,2,-2) find
a) $5\vec{C} - 2\vec{D}$ b) If $3\bar{A} - 4\bar{D} = \bar{c}$, find $\bar{A}$.

SOLUTION

(a) $5\vec{C} - 2\vec{D}$ = 5(2,-3,1) - 2(0,2,-2) = (10, -15,5) - (0,4,-4) = (10,-19,9)
(b) $3\vec{A} = \vec{C} + 4\vec{D}$ = (2, -3,1) + 4(0,2,-2) = (2, -3,1) + (0,8,-8) = (2,5,-7)
$\therefore \vec{A} = \left(\frac{2}{3}, \frac{5}{3}, \frac{-7}{3}\right)$

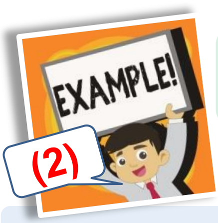
Equality of two vectors:

If $\vec{A} = (A_x, A_y, A_z)$ & $\vec{B} = (B_x, B_y, B_z)$,
Then $\vec{A} = \vec{B}$ If and only If: $A_x = B_x$, $A_y = B_y$, $A_z = B_z$

Find the values of l, m and n which make $\vec{A} = (l - 4, m^2 - 3, 1)$ and $\vec{B}$ = $(5, 1, n^2)$ equal.

SOLUTION

($\vec{A} = \vec{B}$, $l - 4 = 5$, $\therefore l = 9$ & $m^2 - 3 = 1$,$m^2 = 4$, $\therefore m = \pm 2$
$n^2 = 1$, $\therefore n = \pm 1$



if $(2x + 1, 5, k + 4) = (-1, y^2 - 4, x + 1)$ find x, y, k?

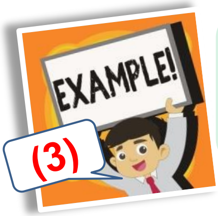
SOLUTION

$2x + 1 = -1$, $2x = -2$, $\therefore x = -1$
$y^2 - 4 = 5$, $y^2 = 9$, $\therefore y = \pm 3$
$k + 4 = x + 1 = 0$, $\therefore k = -4$

* The Unit Vector : Is a vector which its norm equals one length unit.

$\bar{A} = \left(\frac{-3}{13}, \frac{4}{13}, \frac{12}{13}\right)$ is a unit vector as: $\|\bar{A}\| = \sqrt{\frac{9}{169} + \frac{16}{169} + \frac{144}{169}} = \sqrt{\frac{169}{169}}$

$\therefore \|\vec{A}\| = 1$ unit length



Show which of the following vectors represents a unit vector:

$\bar{A} = \left(\frac{2}{3}, \frac{-1}{3}, \frac{-2}{3}\right)$, $\bar{B} = \left(\frac{1}{5}, \frac{4}{5}, \frac{-\sqrt{5}}{5}\right)$

SOLUTION

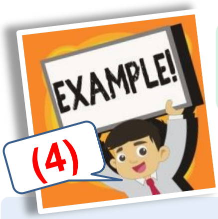
$\|\bar{A}\| = \sqrt{\frac{4}{9} + \frac{1}{9} + \frac{4}{9}} = 1$ Unit length , $\therefore \bar{A}$ is aunit vector

$\|\bar{B}\| = \sqrt{\frac{1}{25} + \frac{16}{25} + \frac{5}{25}} = \sqrt{\frac{22}{25}}$ Unit length , $\therefore \bar{B}$ is not a unit vector

The fundamental unit vectors ($\hat{\imath}, \hat{\jmath}, \hat{k}$):

is a directed segment whose starting point is the origin point and its norm one length unit and its direction is the positive direction of the coordinate axes (x,y,z) respectively.

$\hat{\imath} = (1,0,0)$, $\hat{\jmath} = (0,1,0)$, , $\hat{k} = (0,0,1)$



Write the vectors(-1,0,0), (0,-1,0), (0,0,-1) in terms of the fundamental unit vectors.

SOLUTION

$(-1,0,0) = -(1,0,0) = -\hat{\imath}$, $(0,-1,0) = -(0,1,0) = -\hat{\jmath}$,

$(0,0,-1) = -(0,0,1) = -\hat{k}$

* Expressing the 3D vector in terms of the fundamental unit vectors.

$\bar{A} = (A_x, A_y, A_z) \in R^3$

$\bar{A} = (A_x,0,0) + (0, A_y,0) + (0,0, A_z)$

$\bar{A} = A_x(1,0,0) + A_y(0,1,0) + A_z(0,0,1)$

$\bar{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$

EXAMPLE! (5)

IF $\vec{A}$ = 2$\hat{\imath}$ - 3 $\hat{\jmath}$ + $\hat{k}$, $\vec{B}$ = -$\hat{\imath}$ - 2 $\hat{\jmath}$
Find: a) 2$\vec{A}$- 3$\vec{B}$ b) $\|\vec{A} + \vec{B}\|$. $\|\vec{A}\|$ + $\|\vec{B}\|$. what do you

SOLUTION

$\vec{A}$ = (2, -3,1) , $\vec{B}$ = (-1, -2,0)

a) 2$\vec{A}$ - 3$\vec{B}$ = 2(2, -3,1) - 3(-1, -2,0) = (4, -6,2) + (3,6,0) = (7,0,2) = 7$\hat{\imath}$ + 2$\hat{k}$

b) $\vec{A}$ + $\vec{B}$ = (2, -3,1) + (-1, -2,0) = (1, -5,1)

$\|\vec{A} + \vec{B}\| = \sqrt{1 + 25 + 1} = \sqrt{27}$ Unit length

$\|\vec{A}\| + \|\vec{B}\| = \sqrt{4 + 9 + 1} = \sqrt{1 + 4 + 0} = \sqrt{14} + \sqrt{5}$

deduction: $\sqrt{27} + \sqrt{14} + \sqrt{5}$, $\|\vec{A} + \vec{B}\| \neq \|\vec{A}\| + \|\vec{B}\|$

EXAMPLE! (6)

IF$\vec{A}$ = -3$\hat{\jmath}$ - $\hat{k}$ +5$\hat{\imath}$, $\vec{B}$ = -2$\hat{k}$ + 3$\hat{\imath}$ Find: a) 3$\vec{A}$ - 5$\vec{B}$ b) $\|\vec{A} - \vec{B}\|$

SOLUTION

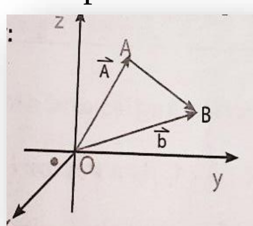
$\vec{A}\vec{A}$ = (5.-3,-1), B = (3,0,-2) a) 3A - 5B = 3(5,-3,-1) - 5(3,0,-2) = (15,-9,-3) - (15,0,-10) = (0, -9,7) = -9 $\hat{\jmath}$ + 7$\hat{k}$

b) $\vec{A}$ - $\vec{B}$ = (5,-3, -1)-(3,0,-2) = (2, -3,1)

$\|\vec{A} - \vec{B}\| = \sqrt{4 + 9 + 1} = \sqrt{14}$ Unit length.

Expressing the directed line segment in the space in terms of the coordinates of its ends.

Let A, B be two points in the space, their position vectors(with respect to the origin point)are $\overrightarrow{OA}$ and $\overrightarrow{OB}$ respectively:



According to the triangle rule $\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$

$\overrightarrow{OA} = \vec{A} - \vec{O} = \vec{A}$ & $\overrightarrow{OB} = \vec{B} - \vec{O} = \vec{B}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \vec{B} - \vec{A}$

EXAMPLE! (7)

IF$\vec{A}$ = (-2,3,1), $\vec{B}$ = (4,0, 2) find $\overrightarrow{AB}$ and $\overrightarrow{BA}$.

SOLUTION

$\vec{A}\overrightarrow{AB} = \vec{B} - \vec{A}$ = (4,0,2) - (-2,3,1) = (6,-3,1)

$\overrightarrow{BA} = \vec{A} - \vec{B}$ = (-2,3,1) - (4,0,2) = (-6,3,-1)