Example (2)

Find : $\int \frac{(1+\ln x)^2}{x}\ dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Step (1) : Let $u = 1+\ln x \Rightarrow du = \frac{1}{x}\ dx \rightarrow dx = x\,du$

$$\therefore\ I = \int \frac{\cancel{x}\,u^2}{\cancel{x}}\ du = \int u^2\ du$$

As you see the integration here is more simpler than the original one

Step (2) : $I = \frac{u^3}{3} + c$ Step (3) : $I = \frac{1}{3}(1+\ln x)^3 + c$

Example (3)

Find : $\int \frac{1}{\sqrt{x}\ e^{\sqrt{x}}}\ dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Step (1) : Let $u = \sqrt{x} \Rightarrow du = \frac{1}{2\sqrt{x}}\ dx \rightarrow dx = 2\sqrt{x}\ du$

$$\therefore\ I = \int \frac{2\cancel{\sqrt{x}}}{\cancel{\sqrt{x}}\ e^u}\ du = \int \frac{2}{e^u}\ du = \int 2\ e^{-u}\ du$$

As you see the integration here is more simpler than the original one

Step (2) : $I = -2e^{-u} + c$ Step (3) : $I = -2e^{-\sqrt{x}} + c$

Example (4)

Find : $\int x^3 (2x^4 - 7)^5\ dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Step (1) : Let $u = 2x^4 - 7 \Rightarrow du = 8x^3\ dx \rightarrow dx = \frac{du}{8x^3}$

$$\therefore\ I = \int \frac{u^5}{8}\ du = \frac{1}{8}\int u^5\ du$$

As you see the integration here is more simpler than the original one

Step (2) : $I = \frac{1}{8} \times \frac{u^6}{6} + c$ Step (3) : $I = \frac{(2x^4 - 4)^6}{48} + c$

Example (5)

Find : $\int \frac{x+4}{\left(x^2+8x\right)^3}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Step (1) : Let $u = x^2+8x \Rightarrow du = 2x+8\ dx \rightarrow dx = \frac{du}{2(x+4)}$

$$\therefore\ I = \int \frac{\cancel{(x+4)}}{2\,\cancel{(x+4)}u^3}\,du = \frac{1}{2}\int u^{-3}\ du$$

As you see the integration here is more simpler than the original one

Step (2) : $I = \frac{1}{2}\times\frac{u^{-2}}{-2}+c \quad \Rightarrow \quad$ Step (3) : $I = \frac{\left(x^2+8x\right)^{-2}}{-4}+c = \frac{-1}{4\left(x^2+8x\right)^2}+c$

Example (6)

Find : $\int \sqrt{x}\ Sin\left(1+\sqrt{x^3}\right)dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Let $u = x^3 \quad \Rightarrow \quad du = 3x^2\ dx \rightarrow dx = \frac{du}{3x^2}$

$$\therefore\ I = \int \frac{\sqrt{x}\ Sin\left(1+\sqrt{u}\right)}{3x^2}\ du \quad \times\times\times\times$$

$\because$ The new function is more complicated than the original function

$\therefore$ You have to change your substitution style

Let $u = \sqrt{x^3} = x^{\frac{3}{2}} \Rightarrow du = \frac{3}{2}x^{\frac{1}{2}}\ dx \Rightarrow du = \frac{3}{2}\sqrt{x}\,dx \Rightarrow dx = \frac{2\ du}{3\ \sqrt{x}}$

$$\therefore\ I = \int \frac{2\cancel{\sqrt{x}}\ Sin(1+u)}{3\cancel{\sqrt{x}}}\ du = \frac{2}{3}\int Sin(1+u)\ du$$

$\therefore\ I = \frac{-2}{3}Cos(1+u)+c \quad \Rightarrow \quad \therefore\ I = \frac{-2}{3}Cos\left(1+\sqrt{x^3}\right)+c$

Example (7)

5min

Find : $\int \frac{1}{x(\ln x)^2}\,dx$

Answer

....

....

....

....

....

....

.... $\boxed{Ans : \frac{-1}{\ln x}+c}$

Example (8)

Find : $\int \sqrt{\frac{1+\sqrt{x}}{x}}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

$$\text{Let } \boxed{\begin{matrix} u=\sqrt{x} \\ x=u^2 \end{matrix}} \Rightarrow du=\frac{1}{2\sqrt{x}}dx \rightarrow dx=2\sqrt{x}\,du=2u\ du$$

$$\therefore\ I=\int\sqrt{\frac{1+u}{u^2}}\times 2u\ du \Rightarrow \therefore\ I=2\int\frac{\sqrt{1+u}}{u}\times u\ du \Rightarrow \therefore\ I=2\int(1+u)^{\frac{1}{2}}\ du$$

$$\therefore\ I=2\left[\frac{2}{3}(1+u)^{\frac{3}{2}}\right]+c \Rightarrow \boxed{\therefore\ I=\frac{4}{3}\left(1+\sqrt{x}\right)^{\frac{3}{2}}+c}$$

Example (9)

Find : $\int x^3 \cos\left(x^4+2\right)\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

$$\text{Let } u=x^4+2 \Rightarrow du=4x^3\,dx \rightarrow dx=\frac{du}{4x^3}$$

$$\therefore\ I=\int \cancel{x^3}\cos u\,\frac{du}{4\,\cancel{x^3}} \Rightarrow \therefore\ I=\frac{1}{4}\int \cos u\ du \Rightarrow \therefore\ I=\frac{1}{4}\sin u+c$$

$$\boxed{\therefore\ I=\frac{1}{4}\sin\left(x^4+2\right)+c}$$

Example (10)

$$\int \sqrt{Cot\,x}\;Cosec^2x\;dx$$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

$$Let\quad u = Cot \qquad du = -Cosec^2x\;dx \;\rightarrow\; dx = \frac{-1}{Cosec^2x}\,du$$

$$\therefore\; I = \int \frac{\sqrt{u}\;\cancel{Cosec^2x}}{-\;\cancel{Cosec^2x}}\,du = -\int \sqrt{u}\;du = -\int u^{\frac{1}{2}}\;du$$

$$\therefore\; I = \frac{-2}{3}u^{\frac{3}{2}} + c \;\Rightarrow\; \therefore I = \frac{-2}{3}Cot^{\frac{3}{2}} + c$$

Example (11)

$$\int \frac{dt}{Cos^2t\sqrt{1+Tant}}$$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

$$Let\quad u = 1+Tant \;\Rightarrow\; du = Sec^2t\;dt \;\rightarrow\; dt = \frac{1}{Sec^2t}\,du = Cos^2t\,du$$

$$\therefore\; I = \int \frac{\cancel{Cos^2t}\,du}{\cancel{Cos^2t}\,\sqrt{u}} = \int \frac{1}{\sqrt{u}}\,du = \int u^{-\frac{1}{2}}\;du \;\Rightarrow\; \therefore\; I = 2u^{\frac{1}{2}} + c \;\Rightarrow\; \therefore I = 2\sqrt{1+Tant} + c$$

Example (12)

$$\int \frac{2^x}{2^x+3}\,dx$$

Answer

$$Let\quad u = 2^x \qquad du = 2^x\ln 2\;dx \;\rightarrow\; dx = \frac{1}{2^x\ln 2}\,du = \frac{du}{u\ln 2}$$

$$\therefore\; I = \int \frac{u}{u+3}\times\frac{du}{u\ln 2} = \frac{1}{\ln 2}\int \frac{\overset{f'(x)}{du}}{\underset{f(x)}{u+3}}\,du$$

$$\therefore\; I = \frac{1}{\ln 2}\ln\left|u+3\right| + c \;\Rightarrow\; \therefore I = \frac{1}{\ln 2}\ln\left|2^x+3\right| + c$$

Another solution

$$I = \frac{1}{\ln 2}\int \frac{\overset{f'(x)}{2^x\ln 2}}{\underset{f(x)}{2^x+3}}\,dx$$

$$\therefore\; I = \frac{1}{\ln 2}\ln\left|2^x+3\right| + c$$

Example (13)

$$\int \frac{Sin\,2x}{1+Cos^2 x}\,dx$$

Answer

$$\because \int \frac{2\,Sin\,x\,Cos\,x}{1+Cos^2 x}\,dx \Rightarrow let \quad u = Cos\,x \qquad du = -Sin\,x\,dx \to dx = \frac{-du}{Sin\,x}$$

$$\therefore \quad I = \int \frac{2\,\cancel{Sin\,x}\,u}{1+u^2} \times \frac{-du}{\cancel{Sin\,x}} = -\int \frac{\overset{f'(x)}{2u}}{\underset{f(x)}{1+u^2}}\,du$$

$$\therefore \quad I = -\ln\left|1+u^2\right| + c \quad \Rightarrow \quad \therefore I = -\ln\left|1+Cos^2 x\right| + c$$

Example (14)

$$\int Sin\,x\,Sec^2\left(Cos\,x\right)\,dx$$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

$$Let \quad u = Cos\,x \qquad du = -Sin\,x\,dx \to dx = \frac{-du}{Sin\,x}$$

$$\therefore \quad I = \int \cancel{Sin\,x}\,Sec^2 u \times \frac{-du}{\cancel{Sin\,x}} = -\int Sec^2 u\,du$$

$$\therefore \quad I = -Tan\,u + c \quad \Rightarrow \quad \therefore I = -Tan\left(Cos\,x\right) + c$$

Example (15)

$$\int Tan\,x\,\ln\left(Cos\,x\right)\,dx$$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

$$Let \quad u = Cos\,x \qquad du = -Sin\,x\,dx \to dx = \frac{-du}{Sin\,x}$$

$$\therefore \quad I = \int \frac{\cancel{Sin\,x}}{Cos\,x}\ln u \times \frac{-du}{\cancel{Sin\,x}} = -\int \frac{\ln u}{u}\,du \quad \Rightarrow \quad \boxed{Can't\ be\ solved\ easily}$$

$$Change\ your\ way: \ let \quad u = \ln\left(Cos\,x\right) \quad \Rightarrow \quad du = \frac{-Sin\,x}{Cos\,x}\,dx = -Tan\,x\,dx \to dx = \frac{-du}{Tan\,x}$$

$$\therefore \quad I = \int u\,\cancel{Tan\,x} \times \frac{-du}{\cancel{Tan\,x}} = -\int u\,du$$

$$\therefore \quad I = -\frac{u^2}{2} + c \quad \Rightarrow \quad \therefore \quad I = -\frac{1}{2}\left[\ln\left(Cos\,x\right)\right]^2 + c$$

Example (16)

Find : $\int \frac{x+1}{\sqrt{x-1}}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Let $\boxed{\begin{array}{l} u = x-1 \\ x = u+1 \end{array}} \quad \Rightarrow \quad du = dx$

$$\therefore\ I = \int \frac{u+1+1}{\sqrt{u}}\,du \ \Rightarrow\ \therefore\ I = \int \frac{u+2}{\sqrt{u}}\,du \ \Rightarrow\ \therefore\ I = \int u^{\frac{1}{2}} + 2u^{\frac{-1}{2}}\ du$$

$$\therefore\ I = \frac{2}{3}u^{\frac{3}{2}} + 4u^{\frac{1}{2}} + c \ \Rightarrow\ \therefore\ I = \frac{2}{3}(x-1)^{\frac{3}{2}} + 4(x-1)^{\frac{1}{2}} + c$$

Try to simplify it as you can :

$$\therefore\ I = \frac{2}{3}(x-1)^{\frac{1}{2}}\left[(x-1)^{\frac{2}{2}} + 6\right] + c \ \Rightarrow\ \therefore\ I = \frac{2(x+5)\sqrt{x-1}}{3} + c$$

Example (17)

Find : $\int \frac{(x+3)^3 - 27}{x}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Let $\boxed{\begin{array}{l} u = x+3 \\ x = u-3 \end{array}} \quad \Rightarrow \quad du = dx$

$$\therefore\ I = \int \frac{u^3 - 27}{u-3}\,du \ \Rightarrow\ \therefore\ I = \int \frac{(u-3)(u^2+3u+9)}{(u-3)}\,du \ \Rightarrow\ \therefore\ I = \int u^2 + 3u + 9\ du$$

$$\therefore\ I = \frac{1}{3}u^3 + \frac{3}{2}u^2 + 9u + c \ \Rightarrow\ \therefore\ I = \frac{1}{3}(x+3)^3 + \frac{3}{2}(x+3)^2 + 9(x+3) + c$$

Try to simplify it as you can :

$$\therefore\ I = \frac{(x+3)}{3\times 2}\left[2(x+3)^2 + 9(x+3) + 9\times 3\times 2\right] + c$$

$$\therefore\ I = \frac{(x+3)}{6}\left[2x^2 + 12x + 18 + 9x + 27 + 54\right] + c \ \Rightarrow\ \therefore\ I = \frac{(x+3)}{6}\left[2x^2 + 21x + 99\right] + c$$

Example (18)

Find : $\int \frac{x^3}{\sqrt{x^2+1}}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Let $\boxed{\begin{array}{l} u = x^2+1 \\ x^2 = u-1 \end{array}} \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$$\therefore\ I = \int \frac{x^3}{\sqrt{u}}\frac{du}{2x} \Rightarrow \therefore\ I = \frac{1}{2}\int \frac{x^2}{\sqrt{u}}\,du \Rightarrow \therefore\ I = \frac{1}{2}\int \frac{u-1}{\sqrt{u}}\,du \Rightarrow \therefore\ I = \frac{1}{2}\int u^{\frac{1}{2}} - u^{\frac{-1}{2}}\,du$$

$$\therefore\ I = \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right] + c \Rightarrow \therefore\ I = \frac{1}{2}\left[\frac{2}{3}\left(x^2+1\right)^{\frac{3}{2}} - 2\left(x^2+1\right)^{\frac{1}{2}}\right] + c$$

$$\therefore\ I = \frac{2}{2}\left[\frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} - \left(x^2+1\right)^{\frac{1}{2}}\right] + c \Rightarrow \therefore\ I = \frac{1}{3}\left(x^2+1\right)^{\frac{3}{2}} - \left(x^2+1\right)^{\frac{1}{2}} + c$$

Try to simplify it as you can : $\therefore\ I = \frac{\left(x^2+1\right)^{\frac{1}{2}}}{3}\left[\left(x^2+1\right)^{\frac{2}{2}} - 3\right] + c$

$$\therefore\ I = \frac{\sqrt{x^2+1}}{3}\left[x^2+1-3\right] + c \Rightarrow \boxed{\therefore\ I = \frac{\sqrt{x^2+1}}{3}\left[x^2-2\right] + c}$$

Example (19)

Find : $\int x^3\left(x^2-1\right)^5\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Let $\boxed{\begin{array}{l} u = x^2-1 \\ x^2 = u+1 \end{array}} \Rightarrow du = 2x\,dx \Rightarrow dx = \frac{du}{2x}$

$$\therefore\ I = \int x^2\left(x^2-1\right)^5 \times x\,dx \Rightarrow \therefore\ I = \int \frac{(u+1)u^5\cancel{x}}{2\cancel{x}}\,du \Rightarrow \therefore\ I = \frac{1}{2}\int (u+1)u^5\,du$$

$$\therefore\ I = \frac{1}{2}\int u^6 + u^5\,du \Rightarrow \therefore\ I = \frac{1}{2}\left[\frac{u^7}{7} + \frac{u^6}{6} + c\right] \Rightarrow \therefore\ I = \frac{1}{2}\left[\frac{\left(x^2-1\right)^7}{7} + \frac{\left(x^2-1\right)^6}{6}\right] + c$$

Try to simplify it as you can : $\therefore\ I = \frac{\left(x^2-1\right)^6}{2\times(7\times6)}\left[6\left(x^2-1\right)+7\right] + c$

$$\therefore\ I = \frac{\left(x^2-1\right)^6}{84}\left[6x^2-6+7\right] + c \Rightarrow \boxed{\therefore\ I = \frac{1}{84}\left(x^2-1\right)^6\left[6x^2+1\right] + c}$$

Example (20)

Find : $\int x^2 \sqrt[3]{3x+1}\;dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Let $\boxed{\begin{array}{l} u = 3x+1 \\ x = \dfrac{u-1}{3} \end{array}} \quad \Rightarrow \; du = 3\,dx \;\rightarrow\; dx = \dfrac{du}{3}$

$$\therefore\; I = \int \left[\frac{u-1}{3}\right]^2 \sqrt[3]{u}\;\frac{du}{3} = \int \left(\frac{u^2-2u+1}{9}\right) u^{\frac{1}{3}}\;\frac{du}{3} = \frac{1}{27}\int \left(u^2-2u+1\right)u^{\frac{1}{3}}\;du$$

$$\therefore\; I = \frac{1}{27}\int u^{\frac{7}{3}} - 2u^{\frac{4}{3}} + u^{\frac{1}{3}}\;du \quad \Rightarrow \quad \therefore\; I = \frac{1}{27}\left[\frac{3}{10}u^{\frac{10}{3}} - 2\times\frac{3}{7}u^{\frac{7}{3}} + \frac{3}{4}u^{\frac{4}{3}}\right]$$

$$\boxed{\therefore\; I = \frac{1}{27}\left[\frac{3}{10}(3x+1)^{\frac{10}{3}} - \frac{6}{7}(3x+1)^{\frac{7}{3}} + \frac{3}{4}(3x+1)^{\frac{4}{3}}\right] + c}$$

Try to simplify it as you can :

$$\therefore\; I = \frac{1}{27\times(10\times7\times4)}\left[(3\times7\times4)(3x+1)^{\frac{10}{3}} - (6\times10\times4)(3x+1)^{\frac{7}{3}} + (3\times10\times7)(3x+1)^{\frac{4}{3}}\right] + c$$

$$\therefore\; I = \frac{1}{7560}\left[84(3x+1)^{\frac{10}{3}} - 240(3x+1)^{\frac{7}{3}} + 210(3x+1)^{\frac{4}{3}}\right] + c$$

$$\therefore\; I = \frac{6(3x+1)^{\frac{4}{3}}}{7560}\left[14(3x+1)^{\frac{6}{3}} - 40(3x+1)^{\frac{3}{3}} + 35\right] + c$$

$$\therefore\; I = \frac{(3x+1)^{\frac{4}{3}}}{1260}\left[14\left(9x^2+6x+1\right) - 40(3x+1) + 35\right] + c$$

$$\therefore\; I = \frac{(3x+1)^{\frac{4}{3}}}{1260}\left[126x^2 + 84x + 14 - 120x - 40 + 35\right] + c$$

$$\therefore\; I = \frac{(3x+1)^{\frac{4}{3}}}{1260}\left[126x^2 - 36x + 9\right] + c \;\Rightarrow\; \therefore\; I = \frac{9(3x+1)^{\frac{4}{3}}}{1260}\left[14x^2 - 4x + 1\right] + c$$

$$\boxed{\therefore\; I = \frac{(3x+1)^{\frac{4}{3}}\left[14x^2 - 4x + 1\right]}{140} + c}$$

Example (21)

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Find : $\int \left(x^2+5\right)\sqrt{x-1}\ dx$

Answer

Let $\boxed{\begin{array}{l} u = x-1 \\ x = u+1 \end{array}}$ $\Rightarrow\ du = dx$

$$\therefore\ I = \int\left[\left(u+1\right)^2+5\right]\sqrt{u}\ du = \int\left(u^2+2u+1+5\right)u^{\frac{1}{2}}\ du = \int u^{\frac{5}{2}}+2u^{\frac{3}{2}}+6u^{\frac{1}{2}}\ du$$

$$\therefore\ I = \frac{2}{7}u^{\frac{7}{2}}+\frac{4}{5}u^{\frac{5}{2}}+4u^{\frac{3}{2}}+c\ \Rightarrow\ \therefore\ I = \frac{2}{7}\left(x-1\right)^{\frac{7}{2}}+\frac{4}{5}\left(x-1\right)^{\frac{5}{2}}+4\left(x-1\right)^{\frac{3}{2}}+c$$

Try to simplify it as you can : $\therefore\ I = 2\left(x-1\right)^{\frac{3}{2}}\left[\frac{1}{7}\left(x-1\right)^{\frac{4}{2}}+\frac{2}{5}\left(x-1\right)+2\right]+c$

$$\therefore\ I = \frac{2}{7\times5}\left(x-1\right)^{\frac{3}{2}}\left[5\left(x-1\right)^2+\left(2\times7\right)\left(x-1\right)+\left(2\times7\times5\right)\right]+c$$

$$\therefore\ I = \frac{2}{35}\left(x-1\right)^{\frac{3}{2}}\left[5x^2-10x+5+14x-14+70\right]+c$$

$$\boxed{\therefore\ I = \frac{2}{35}\sqrt{\left(x-1\right)^3}\left[5x^2+4x+61\right]+c}$$

Example (22)

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Find : $\int\sqrt{1+x^2}\ x^5\ dx$

Answer

Let $\boxed{\begin{array}{l} u = 1+x^2 \\ x^2 = u-1 \end{array}}$ $\Rightarrow\ du = 2x\,dx\quad\Rightarrow\ dx = \dfrac{du}{2x}$

$$\therefore\ I = \int\sqrt{u}\ x^5\times\frac{du}{2x}\ \Rightarrow\ \therefore\ I = \frac{1}{2}\int\sqrt{u}\ x^4\,du\ \Rightarrow\ \therefore\ I = \frac{1}{2}\int\sqrt{u}\left(u-1\right)^2du$$

$$\therefore\ I = \frac{1}{2}\int u^{\frac{1}{2}}\left(u^2-2u+1\right)du\ \Rightarrow\ \therefore\ I = \frac{1}{2}\int u^{\frac{5}{2}}-2u^{\frac{3}{2}}+u^{\frac{1}{2}}\ du$$

$$\therefore\ I = \frac{1}{2}\left[\frac{2}{7}u^{\frac{7}{2}}-2\times\frac{2}{5}u^{\frac{5}{2}}+\frac{2}{3}u^{\frac{3}{2}}\right]+c\ \Rightarrow\ \therefore\ I = \frac{1}{2}\left[\frac{2}{7}\left(1+x^2\right)^{\frac{7}{2}}-\frac{4}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{2}{3}\left(1+x^2\right)^{\frac{3}{2}}\right]+c$$

$$\boxed{\therefore\ I = \frac{1}{7}\left(1+x^2\right)^{\frac{7}{2}}-\frac{2}{5}\left(1+x^2\right)^{\frac{5}{2}}+\frac{1}{3}\left(1+x^2\right)^{\frac{3}{2}}+c}$$

Try to simplify it as you can : $\therefore\ I = \dfrac{\left(1+x^2\right)^{\frac{3}{2}}}{\left(7\times5\times3\right)}\left[15\left(1+x^2\right)^{\frac{4}{2}}-\left(2\times7\times3\right)\left(1+x^2\right)^{\frac{2}{2}}+35\right]+c$

$$\boxed{\therefore\ I = \frac{\left(1+x^2\right)^{\frac{3}{2}}}{105}\left[15x^4-12x^2+8\right]+c}$$

2 LESSON

2^{nd} Technique

Integration by parts

Introduction We use this technique if the substitution method is failed to solve the Integration .

Rules

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Proof We know that: $\frac{d}{dx}\left[f(x)\times g(x)\right] = f(x)g'(x) + g(x)f'(x)$

$$\therefore \int f(x)g'(x)\ dx + \int g(x)f'(x)\ dx = f(x)\times g(x)$$

By arranging the equation: $\therefore \int f(x)g'(x)\ dx = f(x)\times g(x) - \int g(x)f'(x)\ dx$

So if we put: $f(x) = u$ and $g(x) = v \Rightarrow \therefore \int u\,dv = uv - \int v\,du$

Examples

Example (1)

Find : $\int xe^{x}\ dx$

Answer

Let $U = x$ $\qquad dv = e^{x}\ dx$

$du = dx \qquad \boxed{-} \qquad V = e^{x}$

$$\therefore\ I = xe^{x} - \int e^{x}\,dx = xe^{x} - e^{x} + c$$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Example (2)

Find : $\int x\,\mathrm{Sin}\,x\ dx$

Answer

Let $U = x$ $\qquad dv = \mathrm{Sin}\,x\ dx$

$du = dx \qquad \boxed{-} \qquad V = -\mathrm{Cos}\,x$

$$\therefore\ I = -x\,\mathrm{Cos}\,x + \int \mathrm{Cos}\,x\,dx = -x\,\mathrm{Cos}\,x + \mathrm{Sin}\,x + c$$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Example (3)

Find : $\int \frac{x}{e^{2x}}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

Let
$$\begin{tikzcd} U = e^{-2x} \arrow[dr] & dv = x\,dx \\ du = -2e^{-2x}\,dx & V = \frac{1}{2}x^2 \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore \quad I = \frac{1}{2}x^2 e^{-2x} + \boxed{\int \frac{x^2}{e^{-2x}}dx}$$

No direct rule

The aim of using integration by parts is to obtain a simpler integral than the one we started with $\Rightarrow$ and $\because$ $\boxed{\int \frac{x^2}{e^{-2x}}dx}$ is more complicated than the original function

So, change your assumption

Let
$$\begin{tikzcd} U = x \arrow[dr] & dv = e^{-2x}\,dx \\ du = dx & V = \frac{-1}{2}e^{-2x} \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$\therefore \quad I = \frac{-1}{2}xe^{-2x} + \frac{1}{2}\boxed{\int e^{-2x}\,dx}$ $\Rightarrow$ as you see $\boxed{\int e^{-2x}\,dx}$ is more simpler than the original function $\Rightarrow$ $\boxed{\therefore \quad I = \frac{-1}{2}xe^{-2x} - \frac{1}{4}e^{-2x} + c = \frac{-1}{4}e^{-2x}\left[2x+1\right] + c}$

Example (4)

Find : $\int \frac{xe^x}{(x+1)^2}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

Let
$$\begin{tikzcd} U = xe^x \arrow[dr] & dv = (x+1)^{-2}\,dx \\ du = \left(xe^x + e^x\right)dx & V = -(x+1)^{-1} \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore \quad I = \frac{-xe^x}{x+1} + \int \frac{\left(xe^x + e^x\right)}{x+1}dx \quad \Rightarrow \quad \therefore \quad I = \frac{-xe^x}{x+1} + \int \frac{e^x(x+1)}{x+1}dx$$

$$\therefore \quad I = \frac{-xe^x}{x+1} + \int e^x\,dx \quad \Rightarrow \quad \therefore \quad I = \frac{-xe^x}{x+1} + e^x + c$$

$$\therefore \quad I = \frac{-xe^x}{x+1} + \frac{(x+1)e^x}{x+1} + c \quad \Rightarrow \quad \therefore \quad I = \frac{-xe^x + (x+1)e^x}{x+1} + c$$

$$\therefore \quad I = \frac{e^x\left[-x+x+1\right]}{x+1} + c \quad \Rightarrow \quad \boxed{\therefore \quad I = \frac{e^x}{x+1} + c}$$

Example (5)

Find : $\int x\,Sec^2 2x\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

Let
$$\begin{tikzcd} U = x \arrow[dr] & dv = Sec^2 2x\,dx \\ du = dx & V = \frac{1}{2}Tan\,2x \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore\ I = \frac{1}{2}x\,Tan\,2x - \frac{1}{2}\int Tan\,2x\,dx \Rightarrow \therefore\ I = \frac{1}{2}x\,Tan\,2x - \frac{1}{2}\int \frac{Sin\,2x}{Cos\,2x}dx$$

$$\therefore\ I = \frac{1}{2}x\,Tan\,2x + \frac{1}{2\times 2}\int \frac{\overset{f'(x)}{-2Sin\,2x}}{\underset{f(x)}{Cos\,2x}}dx \Rightarrow \therefore\ I = \frac{1}{2}x\,Tan\,2x + \frac{1}{4}\ln\left|Cos\,2x\right| + c$$

$$\boxed{\therefore\ I = \frac{1}{2}x\,Tan\,2x - \frac{1}{4}\ln\left|Sec\,2x\right| + c}$$

Example (6)

Find : $\int x\ln x\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

Let $U = x$ $\quad dv = \ln x\,dx$

$du = dx$ $\quad V = ??? \Rightarrow$ **No direct rule**

Change your assumption :

Let
$$\begin{tikzcd} U = \ln x \arrow[dr] & dv = x\,dx \\ du = \frac{1}{x} & V = \frac{1}{2}x^2 \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore\ I = \frac{1}{2}x^2\ln x - \frac{1}{2}\int \frac{x^2}{x}dx \Rightarrow \therefore\ I = \frac{1}{2}x^2\ln x - \frac{1}{2}\int x\,dx$$

$$\therefore\ I = \frac{1}{2}x^2\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + c \Rightarrow \therefore\ I = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + c$$

$$\boxed{\therefore\ I = \frac{1}{4}x^2\left[2\ln x - 1\right] + c}$$

Example (7)

Find : $\int \operatorname{Cos} x \ln(\operatorname{Sin} x)\, dx$

Answer

$$\text{Let } u = \operatorname{Sin} x \Rightarrow du = \operatorname{Cos} x\, dx \Rightarrow \therefore\ dx = \frac{du}{\operatorname{Cos} x}$$

$$\therefore\ I = \int \cancel{\operatorname{Cos} x} \ln u \frac{du}{\cancel{\operatorname{Cos} x}} \Rightarrow \therefore\ I = \int \ln u\, du$$

No direct rule

So we have to use IBP :

$$\begin{tikzcd} \text{Let } Z = \ln u \arrow[dr] & dy = du \\ dz = \frac{1}{u} du & y = u \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore\ I = u \ln u - \int du \Rightarrow \therefore\ I = u \ln u - u + c \Rightarrow \boxed{\therefore\ I = \operatorname{Sin} x \ln(\operatorname{Sin} x) - \operatorname{Sin} x + c}$$

Example (8)

Find : $\int x^3 e^{x^2}\, dx$

Answer

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

(3) IBP ✓

$$\boxed{\text{Wrong thinking : } \int e^{x^2} dx \neq \frac{e^{x^2}}{2x}}$$

$$\text{Let } u = x^2 \Rightarrow du = 2x\, dx \Rightarrow \therefore\ dx = \frac{du}{2x}$$

$$\therefore\ I = \int u \times x\, e^{u} \frac{du}{2x} \Rightarrow \therefore\ I = \frac{1}{2} \int u\, e^{u}\, du$$

No direct rule

So we have to use IBP :

$$\begin{tikzcd} \text{Let } Z = u \arrow[dr] & dy = e^{u} \\ dz = du & y = e^{u} \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore\ I = \frac{1}{2}\left[u e^{u} - \int e^{u} du\right] \Rightarrow \therefore\ I = \frac{1}{2}\left(u e^{u} - e^{u}\right) + c \Rightarrow \boxed{\therefore\ I = \frac{1}{2}\left[x^2 e^{x^2} - e^{x^2}\right] + c}$$

$$\boxed{\therefore\ I = \frac{e^{x^2}}{2}\left[x^2 - 1\right] + c}$$

Example (9)

Find : $\int (x^2+2x)\,\text{Cos}\,x\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

$$\begin{tikzcd} \text{Let}\quad U = x^2+2x \arrow[dr, "\boxed{-}"'] & dv = \text{Cos}\,x\,dx \\ du = 2x+2\,\,dx & V = \text{Sin}\,x \arrow[l] \end{tikzcd}$$

$\therefore\ I = (x^2+2x)\,\text{Sin}\,x - 2\int \boxed{(x+1)\,\text{Sin}\,x\,dx}$ **No direct rule**

But $\because \boxed{\int (x+1)\,\text{Sin}\,x}$ is more simpler than the original function, then we can integrate it by IBP again :

So let $I_1 = \int (x+1)\,\text{Sin}\,x\,dx \Rightarrow$

$$\begin{tikzcd} \text{Let}\quad U = x+1 \arrow[dr, "\boxed{-}"'] & dv = \text{Sin}\,x\,dx \\ du = dx & V = \text{-Cos}\,x \arrow[l] \end{tikzcd}$$

$\therefore\ I_1 = \text{-}(x+1)\,\text{Cos}\,x + \int \text{Cos}\,x\,dx \quad \Rightarrow \quad \boxed{\therefore\ I_1 = \text{-}(x+1)\,\text{Cos}\,x + \text{Sin}\,x + c}$

Substitute in I: $\therefore I = (x^2+2x)\,\text{Sin}\,x - 2I_1$

$\therefore\ I = (x^2+2x)\,\text{Sin}\,x - 2\left[\text{-}(x+1)\,\text{Cos}\,x + \text{Sin}\,x + c\right]$

$$\boxed{\therefore\ I = (x^2+2x)\,\text{Sin}\,x + 2(x+1)\,\text{Cos}\,x - 2\text{Sin}\,x + c}$$

Example (10)

Find : $\int x^2 e^{x+3}\,dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

$$\begin{tikzcd} \text{Let}\quad U = x^2 \arrow[dr, "\boxed{-}"'] & dv = e^{x+3}\,dx \\ du = 2x\,dx & V = e^{x+3} \arrow[l] \end{tikzcd}$$

$\therefore\ I = x^2 e^{x+3} - 2\int \boxed{x e^{x+3}\,dx}$ **No direct rule**

But $\because \int \boxed{x e^{x+3}\,dx}$ is more simpler than the original function, then we can integrate it by IBP again :

So let $I_1 = \int x e^{x+3}\,dx \Rightarrow$

$$\begin{tikzcd} \text{Let}\quad U = x \arrow[dr, "\boxed{-}"'] & dv = e^{x+3}\,dx \\ du = dx & V = e^{x+3} \arrow[l] \end{tikzcd}$$

$\therefore\ I_1 = x e^{x+3} - \int e^{x+3}\,dx \quad \Rightarrow \quad \boxed{\therefore\ I_1 = x e^{x+3} - e^{x+3} + c}$

Substitute in I: $\therefore I = x^2 e^{x+3} - 2I_1 \quad \Rightarrow \quad \therefore\ I = x^2 e^{x+3} - 2\left[x e^{x+3} - e^{x+3} + c\right]$

$$\boxed{\therefore\ I = x^2 e^{x+3} - 2x e^{x+3} + 2e^{x+3} + c}$$

Example (11)

Find : $\int (x+1)^2 e^{2x}\, dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

$$\begin{tikzcd}
\text{Let } U=(x+1)^2 \arrow[dr] & dv=e^{2x}\, dx \\
du=2(x+1)\, dx & V=\frac{1}{2}e^{2x} \arrow[l, "\boxed{-}"']
\end{tikzcd}$$

$$\therefore\ I=\frac{1}{2}(x+1)^2 e^{2x}-\int \boxed{(x+1)e^{2x}\ dx}$$

No direct rule

But $\because \int \boxed{(x+1)e^{2x}\ dx}$ is more simpler than the original function, then we can integrate it by IBP again :

So let $I_1=\int (x+1)e^{2x}\ dx \Rightarrow$

$$\begin{tikzcd}
\text{Let } U=x+1 \arrow[dr] & dv=e^{2x}\ dx \\
du=dx & V=\frac{1}{2}e^{2x} \arrow[l, "\boxed{-}"']
\end{tikzcd}$$

$$\therefore\ I_1=\frac{1}{2}(x+1)e^{2x}-\frac{1}{2}\int e^{2x}\ dx \ \Rightarrow\ \boxed{\therefore\ I_1=\frac{1}{2}(x+1)e^{2x}-\frac{1}{4}e^{2x}+c}$$

Substitute in I: $\therefore I=\frac{1}{2}(x+1)^2 e^{2x}-I_1$

$$\therefore\ I=\frac{1}{2}(x+1)^2 e^{2x}-\left[\frac{1}{2}(x+1)e^{2x}-\frac{1}{4}e^{2x}+c\right]$$

$$\therefore\ I=\frac{1}{2}(x+1)^2 e^{2x}-\frac{1}{2}(x+1)e^{2x}+\frac{1}{4}e^{2x}+c$$

$$\therefore\ I=\frac{1}{4}e^{2x}\left[2(x+1)^2-2(x+1)+1\right]+c \Rightarrow \therefore\ I=\frac{1}{4}e^{2x}\left[2x^2+4x+2-2x-2+1\right]+c$$

$$\boxed{\therefore\ I=\frac{1}{4}e^{2x}\left[2x^2+2x+1\right]+c}$$

Example (12)

Find : $\int x\,(\ln x)^2\ dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

Let $U=(\ln x)^2$ $\qquad dv = x\,dx$

$du = 2\left(\frac{\ln x}{x}\right)dx \xleftarrow{\boxed{-}} V=\frac{1}{2}x^2$

$$\therefore\ I=\frac{1}{2}x^2(\ln x)^2-\int\boxed{x\ln x\ dx}$$

No direct rule

But $\because \int \boxed{x\ln x\ dx}$ is more simpler than the original function, then we can integrate it by IBP again :

So let $I_1=\int x\ln x\ dx \Rightarrow$

Let $U=\ln x$ $\qquad dv = x\ dx$

$du=\frac{1}{x}dx \xleftarrow{\boxed{-}} V=\frac{1}{2}x^2$

$$\therefore\ I_1=\frac{1}{2}x^2\ln x-\frac{1}{2}\int x\ dx \ \Rightarrow\ \boxed{\therefore I_1=\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c}$$

Substitute in I: $\therefore I=\frac{1}{2}x^2(\ln x)^2-I_1$

$$\therefore\ I=\frac{1}{2}x^2(\ln x)^2-\left[\frac{1}{2}x^2\ln x-\frac{1}{4}x^2+c\right]\Rightarrow\ \therefore\ I=\frac{1}{2}x^2(\ln x)^2-\frac{1}{2}x^2\ln x+\frac{1}{4}x^2+c$$

$$\boxed{\therefore\ I=\frac{1}{4}x^2\left[2(\ln x)^2-2\ln x+1\right]+c}$$

Example (13)

Find : $\int x^3 e^x \; dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

$$\begin{tikzcd}
\text{Let} & U = x^3 \arrow[drr] & & dv = e^x\,dx \\
& du = 3x^2\;dx & & V = e^x \arrow[ll, "\boxed{-}"']
\end{tikzcd}$$

$\therefore\;\; I = x^3 e^x - 3\int \boxed{x^2 e^x \; dx}$ **No direct rule**

But $\because \int \boxed{x^2\, e^x\, dx}$ is more simpler than the original function, then we can integrate it by IBP again :

So let $I_1 = \int x^2\, e^x\, dx \Rightarrow$

$$\begin{tikzcd}
\text{Let} & U = x^2 \arrow[drr] & & dv = e^x\;dx \\
& du = 2x\;dx & & V = e^x \arrow[ll, "\boxed{-}"']
\end{tikzcd}$$

$\therefore\;\; I_1 = x^2\, e^x - 2\int x e^x \; dx$ **No direct rule**

And $\because \int \boxed{x e^x\, dx}$ is more simpler than the previous function, then we can integrate it by IBP once more :

So let $I_2 = \int x e^x\, dx \Rightarrow$

$$\begin{tikzcd}
\text{Let} & U = x \arrow[drr] & & dv = e^x\;dx \\
& du = dx & & V = e^x \arrow[ll, "\boxed{-}"']
\end{tikzcd}$$

$\therefore\;\; I_2 = x e^x - \int e^x \; dx \;\;\Rightarrow\;\; \boxed{\therefore\; I_2 = x e^x - e^x + c}$

Substitute in I_1 : $\;\therefore\;\; I_1 = x^2\, e^x - 2I_2$

$\therefore\;\; I_1 = x^2\, e^x - 2\left[x e^x - e^x + c\right] \;\Rightarrow\; \boxed{\therefore\; I_1 = x^2\, e^x - 2x e^x + 2e^x + c}$

Substitute in I : $\;\therefore I = x^3\, e^x - 3I_1$

$\therefore\;\; I = x^3\, e^x - 3\left[x^2\, e^x - 2x e^x + 2e^x + c\right] \;\Rightarrow\; \boxed{\therefore\; I_1 = x^3\, e^x - 3x^2\, e^x + 6x e^x - 6e^x + c}$

Example (14)

Find : $\int e^x \operatorname{Sin} x \; dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

$$\begin{tikzcd}
\text{Let} \quad U = e^x \arrow[dr] & dv = \operatorname{Sin} x \; dx \\
du = e^x \; dx & V = -\operatorname{Cos} x \arrow[l, "\boxed{-}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\therefore \; I = -e^x \operatorname{Cos} x + \int \boxed{e^x \operatorname{Cos} x \; dx} \arrow[r, bend left] & \textbf{No direct rule}
\end{tikzcd}$$

But $\because \int \boxed{e^x \operatorname{Cos} x \, dx}$ *is not difficult than the original function, then we can integrate it by IBP again :*

So let $I_1 = \int e^x \operatorname{Cos} x \, dx \Rightarrow$

$$\begin{tikzcd}
\text{Let} \quad U = e^x \arrow[dr] & dv = \operatorname{Cos} x \; dx \\
du = e^x \; dx & V = \operatorname{Sin} x \arrow[l, "\boxed{-}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\therefore \; I_1 = e^x \operatorname{Sin} x - \int e^x \operatorname{Sin} x \; dx \arrow[r, bend left] & \textbf{No direct rule}
\end{tikzcd}$$

And $\because \int \boxed{x \operatorname{Sin} x \; dx}$ *is the same as the original function .*

Substitute in I_1 : $\boxed{\therefore \; I_1 = e^x \operatorname{Sin} x - I}$

Substitute in I : $\therefore \; I = -e^x \operatorname{Cos} x + I_1$

$\therefore \; I = -e^x \operatorname{Cos} x + e^x \operatorname{Sin} x - I \;\; \Rightarrow \;\; \therefore \; 2I = -e^x \operatorname{Cos} x + e^x \operatorname{Sin} x$

$$\boxed{\therefore \; I = \frac{-1}{2} e^x \operatorname{Cos} x + \frac{1}{2} e^x \operatorname{Sin} x + c}$$

Example (15)

Find : $\int e^{-x}\cos 2x\ dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ×

(3) IBP ✓

Answer

$$\begin{tikzcd} \text{Let}\quad U = e^{-x} \arrow[dr] & dv = \cos 2x\ dx \\ du = -e^{-x}\ dx & V = \frac{1}{2}\sin 2x \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore\ I = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\int \boxed{e^{-x}\sin 2x\ dx}$$ **No direct rule**

But $\because \int \boxed{e^{-x}\sin 2x\ dx}$ is not difficult than the original function, then we can integrate it by IBP again :

So let $I_1 = \int e^{-x}\sin 2x\ dx \Rightarrow$

$$\begin{tikzcd} \text{Let}\quad U = e^{-x} \arrow[dr] & dv = \sin 2x\ dx \\ du = -e^{-x}\ dx & V = \frac{-1}{2}\cos 2x \arrow[l, "\boxed{-}"'] \end{tikzcd}$$

$$\therefore\ I_1 = \frac{-1}{2}e^{-x}\cos 2x - \frac{1}{2}\int e^{-x}\cos 2x\ dx$$ **No direct rule**

And $\because \int \boxed{e^{-x}\cos 2x\ dx}$ is the same as the original function .

Substitute in I_1 : $\boxed{\therefore\ I_1 = \frac{-1}{2}e^{-x}\cos 2x - \frac{1}{2}I}$

Substitute in I : $\therefore\ I = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}I_1$

$$\therefore\ I = \frac{1}{2}e^{-x}\sin 2x + \frac{1}{2}\left[\frac{-1}{2}e^{-x}\cos 2x - \frac{1}{2}I\right] \Rightarrow \therefore\ I = \frac{1}{2}e^{-x}\sin 2x - \frac{-1}{4}e^{-x}\cos 2x - \frac{1}{4}I$$

$$\therefore\ I + \frac{1}{4}I = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x \Rightarrow \therefore\ \frac{5}{4}I = \frac{1}{2}e^{-x}\sin 2x - \frac{1}{4}e^{-x}\cos 2x$$

$$\left(\text{Multiply by } \frac{4}{5}\right) \Rightarrow \boxed{\therefore\ I = \frac{2}{5}e^{-x}\sin 2x - \frac{1}{5}e^{-x}\cos 2x + c}$$

Example (16)

Find : $\int x\, ln(1+x)\, dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

(3) IBP ✓

Answer

Let $\boxed{\begin{matrix} u = x+1 \\ x = u-1 \end{matrix}} \Rightarrow \quad du = d\,x$

$\therefore \quad I = \int (u-1)\, ln\, u\; du$ **No direct rule**

So we have to use IBP :

$$\boxed{\begin{matrix} \text{Let} \quad Z = ln\, u & dy = u - 1 \\ dz = \frac{1}{u}\, du & y = \frac{1}{2}u^2 - u \end{matrix}}$$

$$\therefore \quad I = \left(\frac{1}{2}u^2 - u\right) ln\, u - \frac{1}{2}\int u - 1\, du \quad \Rightarrow \therefore\, I = \left(\frac{1}{2}u^2 - u\right) ln\, u - \frac{1}{4}u^2 + u + c$$

$$\therefore\, I = \left[\frac{1}{2}(x+1)^2 - (x+1)\right] ln(x+1) - \frac{1}{4}(x+1)^2 + (x+1) + c$$

$$\therefore\, I = (x+1)\left[\frac{1}{2}(x+1) - 1\right] ln(x+1) - \frac{1}{4}(x+1)^2 + (x+1) + c$$

$$\therefore\, I = (x+1)\left[\frac{1}{2}x + \frac{1}{2} - 1\right] ln(x+1) - \frac{1}{4}(x+1)^2 + (x+1) + c$$

$$\therefore\, I = (x+1)\left[\frac{1}{2}x - \frac{1}{2}\right] ln(x+1) - \frac{1}{4}(x+1)^2 + (x+1) + c$$

$$\therefore\, I = \frac{1}{2}(x+1)(x-1)\, ln(x+1) - \frac{1}{4}(x+1)^2 + (x+1) + c$$

$$\boxed{\therefore\, I = \frac{1}{2}(x^2 - 1)\, ln(x+1) - \frac{1}{4}(x+1)^2 + (x+1) + c}$$