Example (14)

A wire of length 64 cm is cut into two portions , the first is bent to form a square , and the second to form a circle . Find the length of each portion if the sum of the surface areas of the square and the circle is minimum .

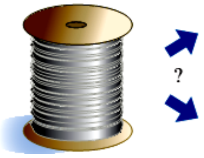
Answer

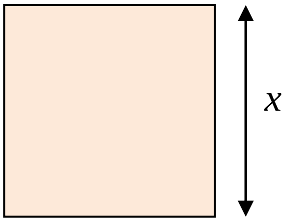
In this problem, we have to make a relation between Area of the two portions and their perimeters

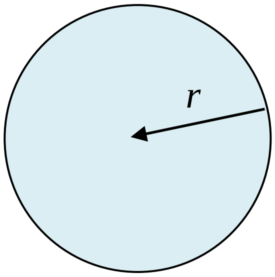
$\because$ *The length of the wire = the length of the two portions = 64 cm*

$\therefore$ *Let the perimeter of the square be $4x$ and the perimeter of the circle be $2\pi r$* $\Rightarrow \therefore\ 4x + 2\pi r = 64 \ (\div 2)$

$$\boxed{\therefore\ x = \frac{1}{2}(32 - \pi r) - - - (1)}$$







And $\text{Area}\ (A) = x^2 + \pi r^2 = \frac{1}{4}(32 - \pi r)^2 + \pi r^2$

$$\therefore\ A(r) = 256 - 16\pi r + \frac{1}{4}\pi^2 r^2 + \pi r^2 \Rightarrow \therefore\ A'(r) = -16\pi + \frac{1}{2}\pi^2 r + 2\pi r$$

When $A' = 0 \Rightarrow \therefore\ -16\pi + \frac{1}{2}\pi^2 r + 2\pi r = 0 \ ((\times 2) \text{ And } (\div \pi))$

$$\therefore\ -32 + \pi r + 4r = 0 \Rightarrow \boxed{r = \frac{32}{\pi + 4} - - - (2)}$$

$A''(r) = \frac{1}{2}\pi^2 + 2\pi > 0 \Rightarrow \therefore$ *area of the two portions are minimum when* $r = \frac{32}{\pi + 4}$

So by substituting (2) *in* (1) : $x = \frac{1}{2}\left(32 - \pi\left(\frac{32}{\pi + 4}\right)\right) = 16 - \frac{16\pi}{\pi + 4} = \frac{16\pi + 64 - 16\pi}{\pi + 4} = \frac{64}{\pi + 4}$

$\therefore$ *The length of the first portion* $P_1 = 4x = \frac{256}{\pi + 4}$

And the length of the second portion $P_2 = 2\pi r = 2\pi \times \frac{32}{\pi + 4} = \frac{64\pi}{\pi + 4}$

Example (15)

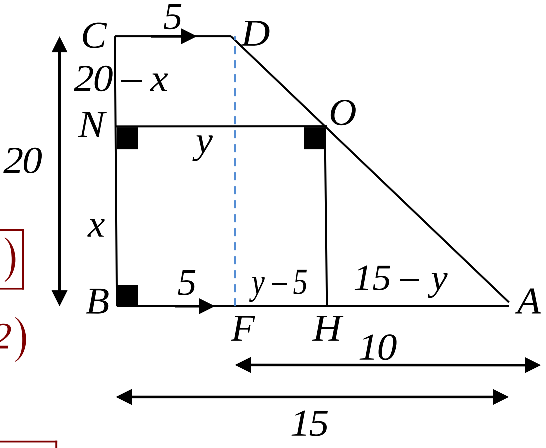
A piece of land in the form of a trapezium ABCD in which $\overline{AB} \parallel \overline{CD}$, $\overline{BC} \perp \overline{AB}$, $AB = 15$ *m* $BC = 20$ *m* , $CD = 5$ *m. It is required to build a house on a portion of this land in the shape of a rectangle so that the point O is taken on* $\overline{AD}$ *and* $\overline{OH}$ *is drawn* $\perp \overline{AB}$ *, and* $\overline{ON} \perp \overline{BC}$ *.*
Find the maximum surface area of the rectangle OHBN .

Answer

> *In this problem, we have to make a relation between area of the rectangle and the sides of trapezuim*

Let the length of the rectangle be x and its width be y

Here we can't make the relation except if we join $\overline{DF}$



To make $\Delta AHO \sim \Delta AFD \Rightarrow \therefore \dfrac{AH}{AF} = \dfrac{HO}{FD}$

$$\frac{15-y}{10} = \frac{x}{20} \Rightarrow 15 - y = \frac{x}{2} \Rightarrow \boxed{\therefore\ x = 30 - 2y ---(1)}$$

And $\because$ $A(\text{area of rectangle}) = x\,y = (30-2y)y ---(2)$

$$\therefore\ A = 30y - 2y^2 \quad \Rightarrow \quad \therefore\ A' = 30 - 4y$$

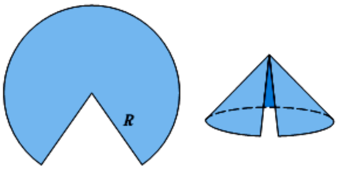
When $A' = 0 \Rightarrow \therefore\ 30 - 4y = 0 \Rightarrow \boxed{y = \dfrac{30}{4} = \dfrac{15}{2} ---(3)}$

And $\because$ $A'' = \text{-}4 < 0 \Rightarrow$ *so the area of the rectangle is maximum at* $y = \dfrac{15}{2}$

Then by substituting in (2) : $\therefore\ A_{Maximum} = (30-2y)y = 15 \times \dfrac{15}{2} = \dfrac{225}{2} = 112.5\ cm^2$.

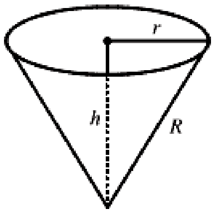
Example (16)

A cone is made from a circular sheet of raduis R by cutting out a sector and gluing the cut edges of the remaining piece together (see the figure)
What is the maximum volume attainable for this cone .



Answer

$$\because V_{cone} = \frac{1}{3}\pi r^2 h \quad \text{and} \quad \boxed{\because\ r^2 = R^2 - h^2} \Rightarrow \therefore\ V = \frac{1}{3}\pi h\left(R^2 - h^2\right) = \frac{\pi}{3}R^2 h - \frac{\pi}{3}h^3$$



$\therefore$ *Differentiate w.r. to h (as R is given)* $\Rightarrow \therefore\ V' = \dfrac{\pi}{3}R^2 - \pi h^2$

When $V' = 0 \Rightarrow \therefore\ \dfrac{\pi}{3}R^2 - \pi h^2 = 0 \Rightarrow h^2 = \dfrac{R^2}{3} \Rightarrow \boxed{\therefore\ h = \dfrac{R}{\sqrt{3}}}$

$V'' = \text{-}2\pi h < 0 \Rightarrow \therefore$ *the maximum volume occurs at* $h = \dfrac{R}{\sqrt{3}}$ *and its value is:*

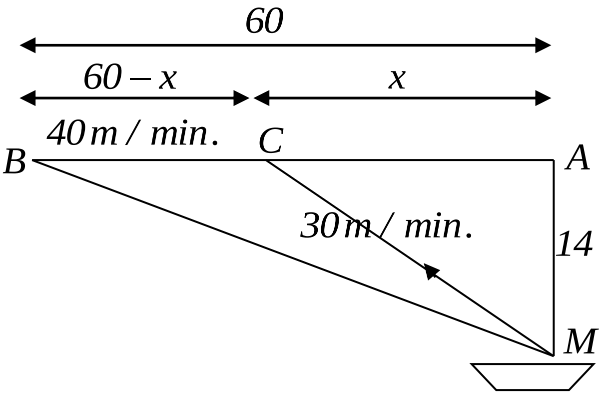
$$V = \frac{\pi}{3}R^2\left(\frac{R}{\sqrt{3}}\right) - \frac{\pi}{3}\left(\frac{R}{\sqrt{3}}\right)^3 = \frac{\pi R^3}{3\sqrt{3}} - \frac{\pi R^3}{9\sqrt{3}} = \frac{2\pi R^3}{9\sqrt{3}}$$

Example (17)

A man in a boat 14 metres from the nearest point A on the straight shore wishes to go to a point B 60 metres down the shore from A. He can land any where on the shore between A and B and walks the rest of the way . He can row with velocity 30 metres / minute and walks with velocity 40 metres / min . Where should he lands in order to reach B in the least time .

Answer

The word nearest here will make me draw the boat perpendicular to A



Also the word least time means that we have to find the time in each part of the problem

So let the boat reaches the shore at C

$\therefore$ The boat will reach point B when it passes through MC then BC will make a time (t) minutes

Where $MC=\sqrt{x^2+196}$ which the boat covers with velocity $30\ m/sec$.

$\therefore$ Its time $T_1=\frac{d}{v}=\frac{\sqrt{x^2+196}}{30}$

$CB=60-x$ which the man covers walking by velocity $40\ m/min.$

$\therefore\ T_2=\frac{d}{v}=\frac{60-x}{40}\ \Rightarrow\ \therefore$ the total time will cover by the boat to reach B is: $T=T_1+T_2$

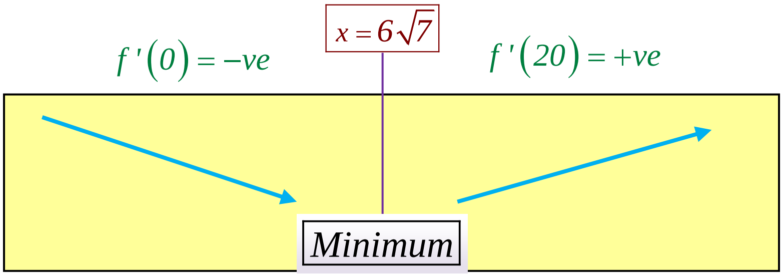
Where $\boxed{t=\frac{1}{30}\sqrt{x^2+196}\ +\ \frac{1}{40}(60-x)---(1)}$

$$\frac{dt}{dx}=\frac{1}{30}\times\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+196}}-\frac{1}{40}\ \Rightarrow\ \therefore\ \text{when}\ \frac{dt}{dx}=0\ \Rightarrow\ \frac{x}{30\sqrt{x^2+196}}-\frac{1}{40}=0$$

$$\therefore\ \frac{x}{30\sqrt{x^2+196}}=\frac{1}{40}\ \Rightarrow\ \therefore\ \frac{x}{\sqrt{x^2+196}}=\frac{3}{4}\ (\text{Square both})\ \Rightarrow\ \therefore\ \frac{x^2}{x^2+196}=\frac{9}{16}$$

$$\therefore\ 16x^2=9x^2+1764\ \Rightarrow\ \therefore\ 7x^2=1764\ \Rightarrow\ \therefore\ x^2=252\ \Rightarrow\ \boxed{\therefore\ x=6\sqrt{7}\ m}$$

Then the least time needed by the boat to reach point B is when the boat land at distance $6\sqrt{7}$ m from A



Example (18)

An iron factory produces two kinds of iron A and B. If the facory produces y tons from A and x tons from B such that $y=\dfrac{40-5x}{10-x}$, where $x\neq 10$, if the price of each ton from kind A is equal twice the price of each ton from kind B. Then how many tons the factory produces from each kind to obtain a maximum profit .

Answer

In this problem, we have to make a relation between the given equation and the price of each ton

$\because$ The factory produces $= A+B=y+x$

Let "C" is the price of each ton from kind "B"

$\therefore$ The total prices of the two kinds per each ton : $P=2C\,y+C\,x=2C\left[\dfrac{40-5x}{10-x}\right]+C\,x$

$$\therefore\ P=\frac{80C-10C\,x}{10-x}+C\,x\ \Rightarrow\ \therefore\ \frac{dP}{dx}=\frac{(10-x)(\text{-}10C)-(80C-10C\,x)(\text{-}1)}{(10-x)^2}+C$$

$$\therefore\ \frac{dP}{dx}=\frac{\text{-}100C+10C\,x+80C-10C\,x}{(10-x)^2}+C=\frac{\text{-}20C}{(10-x)^2}+C=\frac{\text{-}20C+C(10-x)^2}{(10-x)^2}$$

When $\dfrac{dP}{dx}=0\ \Rightarrow\ \therefore\ \text{-}20C+C(10-x)^2=0\ (\div C)\ \Rightarrow\ \therefore\ (10-x)^2=20$

$\therefore\ 10-x=\pm 2\sqrt{5}\ \Rightarrow\ x=10\pm 2\sqrt{5}\ \Rightarrow\ \boxed{\therefore\ x\simeq 5.6\quad \text{Or}\quad x\simeq 14.4}$

And $\because\ \dfrac{d^2P}{dx^2}=\dfrac{\text{-}40C}{(10-x)^3}\ \Rightarrow$ So when $\boxed{x\simeq 5.6\ \text{ton}}\ \Rightarrow\therefore\ \dfrac{d^2P}{dx^2}=\dfrac{\text{-}40C}{(10-x)^3}<0$

Then the profit is maximum and in this case : $y\simeq\dfrac{40-5.6(5)}{10-5.6}\simeq\dfrac{12}{4.4}\simeq 2.7$ tons

And when $\boxed{x\simeq 14.4\ \text{ton}}\ \Rightarrow\therefore\ \dfrac{d^2P}{dx^2}=\dfrac{\text{-}40C}{(10-x)^3}>0$ which is refused in this case

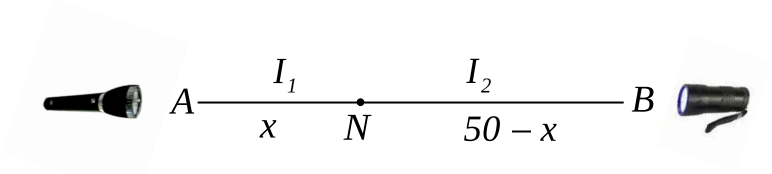
Example (19)

If $AB = 50$ cm and two light sources, one is put at A and the other is put at B. Find the point on $\overline{AB}$ at which the light intensity is minimum, given that the ratio between the light intensities of the two sources is $27 : 8$ and the light intensity at any point is inversely proportional to the square of the distance between the source and this point .

Answer

Let the required point be N , $NA = x \Rightarrow \therefore NB = 50 - x$

And $\because\ I_1\ \alpha\ \frac{1}{x^2} \Rightarrow$ So $I_2\ \alpha\ \frac{1}{(50-x)^2}$



$\therefore\ I_1 = \frac{27k}{x^2} \Rightarrow I_2 = \frac{8k}{(50-x)^2}$

And $\because$ point N is affected by the two intensity light sources

$$\therefore\ I_N = \frac{27k}{x^2} + \frac{8k}{(50-x)^2} = 27k\,x^{-2} + 8k(50-x)^{-2}$$

$$\therefore\ I_N' = -54k\,x^{-3} - 16k(50-x)^{-3}(-1) = \frac{-54k}{x^3} + \frac{16k}{(50-x)^3}$$

And $\because$ I is minimum when $I_N' = 0 \Rightarrow \frac{-54k}{x^3} + \frac{16k}{(50-x)^3} = 0$

$$\therefore\ \frac{16k}{(50-x)^3} = \frac{54k}{x^3} \Rightarrow \frac{8}{(50-x)^3} = \frac{27}{x^3} \Rightarrow \left(\frac{2}{50-x}\right)^3 = \left(\frac{3}{x}\right)^3$$

$$\therefore\ \frac{2}{50-x} = \frac{3}{x} \Rightarrow \therefore\ 2x = 150 - 3x \Rightarrow \therefore\ 5x = 150 \Rightarrow \boxed{\therefore\ x = 30\text{ cm}}$$

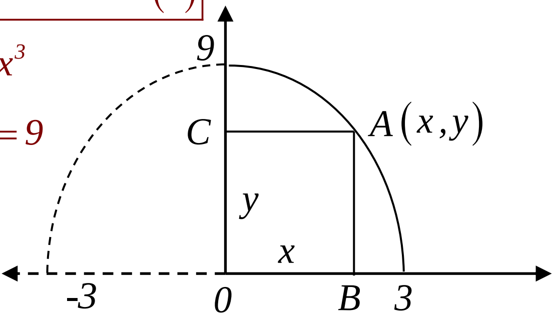
Example (20)

A function f is defined by $f(x) = 9 - x^2$ where $0 \le x \le 3$. A is a point on the curve of f , O is the origin point. $\overline{AB}$ and $\overline{AC}$ are perpendiculars drawn from A to the x - axis and y - axis respectively. Find the coordinates of A in order that the surface area of the rectangle ABOC should be maximum

Answer

> In this problem, we have to make a relation between the required point and the area of the rectangle

Let the dimension of the rectangle be x and y $\Rightarrow \boxed{\because\ y = 9 - x^2 ---(1)}$



And $\because$ the area of the rectagle $A = x\,y = x(9 - x^2) = 9x - x^3$

$\therefore\ A' = 9 - 3x^2 \Rightarrow$ when $A' = 0 \Rightarrow 9 - 3x^2 = 0 \Rightarrow 3x^2 = 9$

$\therefore\ x^2 = 3 \Rightarrow$ so, either $\boxed{x = \sqrt{3}}$ or $\boxed{x = -\sqrt{3}\ (\text{refused})}$

$\therefore\ y = 9 - (\sqrt{3})^2 = 6$, then the coordinate of A is $(\sqrt{3}, 6)$

Example (21)

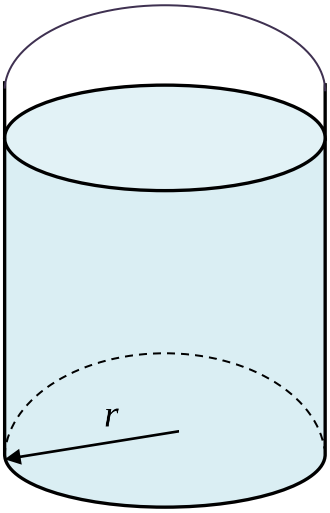
A water tank is in the shape of a cylinder above which there is a semi - sphere such that the upper base of the cylinder is itself the plane surface of the semi - sphere. If the total surface area of the tank equals $20\pi a^2$. Prove that if the volume of the tank is maximum, then the length of the base radius of the cylinder equals its height .

Answer

$\because$ The total area of the cylinder = Area of the two bases + (perimeter of the base × height)

$$\therefore \ \pi r^2 + 2\pi r h + \frac{1}{2} \times 4\pi r^2 = 20\pi a^2 \ \Rightarrow \ \therefore \ 20a^2 = 3r^2 + 2rh$$

$$\boxed{\therefore \ h = \frac{20a^2 - 3r^2}{2r} - - - (1)}$$



$\because$ Volume of the total cylinder :

Area of the base × height + Volume of the semi - sphere

$$\therefore V = \pi r^2 h + \frac{1}{2} \times \frac{4}{3}\pi r^3 = \pi r^2 h + \frac{2}{3}\pi r^3 - - - (2)$$

By substituting (1) in (2) $\therefore \ V = \pi r^2 \left[\frac{20a^2 - 3r^2}{2r} \right] + \frac{2}{3}\pi r^3$

$$\therefore \ V = \frac{\pi}{2}\left(20a^2 r - 3r^3\right) + \frac{2}{3}\pi r^3 \ \Rightarrow \ V' = \frac{\pi}{2}\left(20a^2 - 9r^2\right) + 2\pi r^2$$

And $\because$ the volume is maximum when $\frac{\pi}{2}\left(20a^2 - 9r^2\right) + 2\pi r^2 = 0$

$$\therefore \ 10a^2 = \frac{5}{2}r^2 \ \Rightarrow \ r^2 = 10a^2 \times \frac{2}{5} = 4a^2 \ \Rightarrow \ \boxed{\therefore \ r = 2a}$$

From (1) : $h = \frac{20a^2 - 12a^2}{4a} = \frac{8a^2}{4a} = 2a \ \Rightarrow \ \boxed{\therefore \ r = h}$

Example (22)

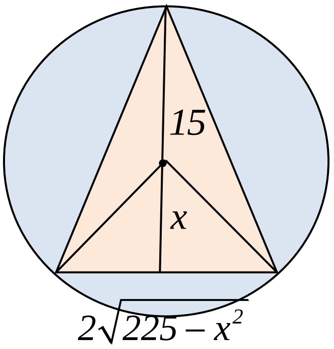
Find the greatest area of an isosceles triangle can be drawn inside a circle of raduis 15 cm

Answer

Let x be the distance between the center of the circle and the base of the isosceles triangle, then the base of the triangle $2\sqrt{225-x^2}$ *and the height of the triangle is* $15+x$. *therefore, the area of the triangle is given by*

$$\because\ A=\frac{1}{2}\left(2\sqrt{225-x^2}\right)(15+x)$$

$$\boxed{\therefore\ A=(15+x)\left(225-x^2\right)^{\frac{1}{2}}\ ---(1)}$$



$$\therefore\ A'=(15+x)\times\frac{1}{2}\left(225-x^2\right)^{\frac{-1}{2}}(-2x)+\left(225-x^2\right)^{\frac{1}{2}}$$

$$\therefore\ \ A'=\sqrt{225-x^2}-\frac{x(15+x)}{\sqrt{225-x^2}}\ \ \text{so, when}\ \ A'=0$$

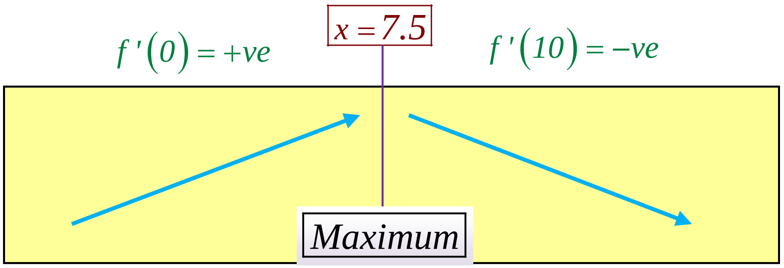
$$\therefore\ \sqrt{225-x^2}-\frac{x(15+x)}{\sqrt{225-x^2}}=0\ \ \left(\text{multiply by }\sqrt{225-x^2}\right)$$

$$\therefore\ 225-x^2-\left(x(15+x)\right)=0\ \Rightarrow\ 225-x^2-15x-x^2=0$$

$$\therefore\ 2x^2+15x-225=0\ \Rightarrow\ (2x-15)(x+15)=0\ \ \Rightarrow\ \boxed{\therefore x=7.5}$$

$\therefore$ *The greatest area of isosceles triangle that can be drawn inside a circle of radius 15 occurs when* $x=7.5$

Then by substituting in (1):

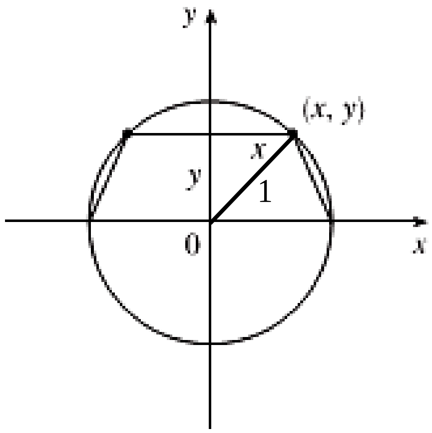


$$A=(15+7.5)\left(225-(7.5)^2\right)^{\frac{1}{2}}=292.28\ cm^2$$

Example (23)

Find the area of the largest trapezoid that can be inscribed in a circle of radius 1 unit and whose base is the diameter of the circle .

Answer



The area of the trapezoid is given by $A=\frac{1}{2}(B_1+B_2)h$

By assuming that $h=y$, $B_1=2$ and $B_2=2x$

$$\boxed{\therefore A=\frac{1}{2}(2+2x)y=y(x+1)---(1)}$$

But $\because\ x^2+y^2=1 \Rightarrow \boxed{\therefore\ y=(1-x^2)^{\frac{1}{2}}---(2)}$

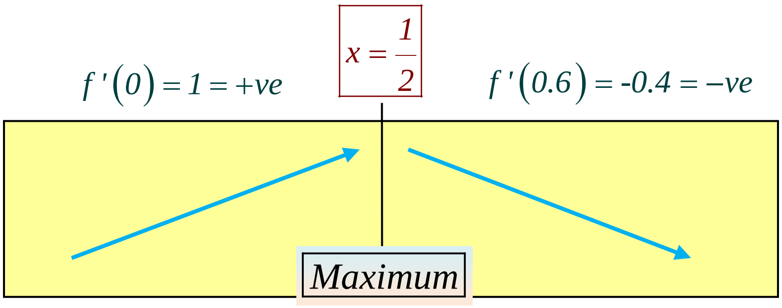
Then substitute (2) in (1): $A=(1-x^2)^{\frac{1}{2}}(x+1)$

$\therefore\ A'=(1-x^2)^{\frac{1}{2}}+\frac{1}{2}(x+1)(1-x^2)^{-\frac{1}{2}}(\text{-}2x) \Rightarrow \boxed{\therefore\ A'=\sqrt{1-x^2}-\frac{x(x+1)}{\sqrt{1-x^2}}}$, then when $A'=0$

$\therefore\ \sqrt{1-x^2}-\frac{x(x+1)}{\sqrt{1-x^2}}=0$ $\left(\text{multiply by } \sqrt{1-x^2}\right)$ $\Rightarrow \therefore 1-x^2-x^2-x=0$

$\therefore\ 2x^2+x-1=0 \Rightarrow (2x-1)(x+1)=0 \Rightarrow \boxed{\therefore\ x=\frac{1}{2}}$ and $\boxed{x=\text{-}1\ (\text{refused})}$

To get A'' :



Then the maximum area of the trapezoid occurs when $x=\frac{1}{2}$ and its value is :

is $A=\left(1-\left(\frac{1}{2}\right)^2\right)^{\frac{1}{2}}\left(\frac{1}{2}+1\right)=\frac{3\sqrt{3}}{4}$ square unit

Example (24)

The cost of the fuel consumption of a locomotive is proportional to the square of its speed. The cost of the fuel consumption is 25 pounds per hour when the speed is 25 km / hr. There is also an additional cost 100 pounds per hour regardless of the locomotive speed . Find the speed of the locomotive which minimizes the total cost of one kilometer .

Answer

Let the cost per hr. is $T \Rightarrow \therefore\ T \,\alpha\, v^2 \rightarrow T = k v^2 \Rightarrow \therefore\ 25 = k(25)^2$

$\therefore\ k = \frac{1}{25} \Rightarrow \therefore\ \boxed{T = \frac{1}{25}v^2 \ \ L.E/hr}$

$\because$ There is an additional cost 100 per hour : $\boxed{\therefore\ C = \frac{1}{25}v^2 t + 100t}$

$\because\ Time = \frac{distance\ (D)}{velocity\ (V)} \Rightarrow after\ (1\ km) \Rightarrow \boxed{\therefore\ T = \frac{1}{V}\ hr}$

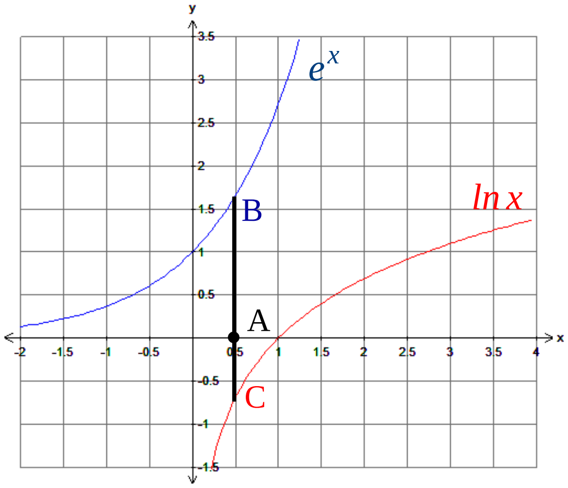
$\therefore\ C = \frac{1}{25}v^2 \times \frac{1}{v} + 100 \times \frac{1}{v} = \frac{1}{25}v + \frac{100}{v} \Rightarrow \therefore\ \frac{dc}{dv} = \frac{1}{25} - \frac{100}{v^2}$

And $\because$ C is minimum when $\frac{dc}{dv} = 0 \Rightarrow \therefore\ \frac{1}{25} - \frac{100}{V^2} = 0 \Rightarrow \therefore\ \frac{100}{V^2} = \frac{1}{25}$

$\therefore\ V^2 = 2500 \Rightarrow \boxed{\therefore\ V = 50\ km/hr}$

Example (25)

In the opposite figure: a point A is moving in the positive direction of x - axis starting from $(0,0)$, find the value of x such that the distance between point B (on the natural exponential curve) and point C (on the natural logarithmic curve) are smallest as possible .



Answer

Let S is the distance between the two curve where :

$S = e^x - \ln x \Rightarrow \therefore\ S' = e^x - \frac{1}{x} \Rightarrow$ when $S' = 0 \Rightarrow \therefore\ e^x = \frac{1}{x} \Rightarrow \boxed{\therefore\ e^x = x^{-1}} \Rightarrow$ ln both sides

$\therefore\ \ln e^x = \ln x^{-1} \Rightarrow \therefore\ x \ln e = -\ln x \Rightarrow \boxed{\therefore\ \ln x = -x} \Rightarrow$ can't be solved except by calculator

$\therefore\ \boxed{\ln x} \rightarrow \boxed{Alpha} \rightarrow \boxed{Calc} \rightarrow \boxed{-x} \rightarrow \boxed{Shift} \rightarrow \boxed{Calc} \rightarrow \boxed{\therefore\ x \simeq 0.567}$

$\because\ S'' = e^x + \frac{1}{x^2} > 0$ [minimum at $x = 0.567$] $\therefore$ the value of x is $\simeq 0.567$

Revision on the previous year on Integration

Remember that Integration is the undo or the reverse of differentiation.

Rules

$$(1)\ \int x^{n}\ dx = \frac{x^{n+1}}{n+1} + c\ ,\ \text{where } n \neq -1\ ,\quad \text{and } c \text{ is a constant}$$

In words: Add (1) to the power and divide by the new power

$$(2)\ \int dx = \int 1\ dx = x + c$$

$$(3)\ \int (ax+b)^{n}\ dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

In words: $\dfrac{\text{Add } (1) \text{ to the power}}{(\text{New power})(\text{derivative between the bracket})}$

Note: There is no general rule for finding the integration of multiplication or division of two functions.

Examples

Find the integral of each of the following functions :

$$(1)\ \int 3x^{5}\ dx$$

Answer

$$\int 3x^{5}\ dx = \frac{3x^{5+1}}{5+1} + c = \frac{3x^{6}}{6} + c = \frac{x^{6}}{2} + c$$

$$(2)\ \int (3x^{2} - 4x + 5)dx$$

Answer

$$\frac{3x^{3}}{3} - \frac{4x^{2}}{2} + 5x = x^{3} - 2x^{2} + 5x + c$$

$$(3)\ \int \frac{5}{2}\ x^{\frac{1}{4}}\ dx$$

Answer

$$\int \frac{5}{2}\ x^{\frac{1}{4}}\ dx = \frac{5}{2} \times \frac{4}{5} x^{\frac{5}{4}} = 2x^{\frac{5}{4}} + c$$

$$(4)\ \int \left(x\sqrt{x} + 2\sqrt{x} - \frac{1}{2\sqrt{x}} \right) \sqrt{x}\ dx$$

Answer

$$\int \left(x^{2} + 2x - \frac{1}{2} \right) dx = \frac{1}{3}x^{3} + x^{2} - \frac{1}{2}x + c$$

$$(5)\ \int (2x+5)^{3}\ dx$$

Answer

$$\frac{(2x+5)^{4}}{2(4)} + c = \frac{1}{8}(2x+5)^{4} + c$$

$$(6)\ \int (3-5x)^{6}\ dx$$

Answer

$$\frac{(3-5x)^{7}}{-5(7)} + c = -\frac{1}{35}(3-5x)^{7} + c$$

Integration of basic trigonometric functions

A Function	Its Integration
(1) If $f(x) = Sin\ x$	$\int Sin\,x\ dx = -Cos\ x + c$
(2) If $f(x) = Cos\ x$	$\int Cos\,x\ dx = Sin\ x + c$
(3) If $f(x) = Sec^2\ x$	$\int Sec^2 x\ dx = Tan\ x + c$
(4) If $f(x) = Sin\ (ax+b)$	$\int Sin\ (ax+b)\ dx = \frac{-Cos(ax+b)}{a} + c$
(5) If $f(x) = Cos(ax+b)$	$\int Cos(ax+b)\ dx = \frac{Sin\ (ax+b)}{a} + c$
(6) If $f(x) = Sec^2(ax+b)$	$\int Sec^2(ax+b)\ dx = \frac{Tan\ (ax+b)}{a} + c$

Note Remember the following relations between the trigonometric functions

(1) $\because\ Sin^2 x + Cos^2 x = 1 \ \Rightarrow\ \therefore\ Cos^2 x = 1 - Sin^2 x$ or $Sin^2 x = 1 - Cos^2 x$

(2) $\because\ 1 + Tan^2 x = Sec^2 x \ \Rightarrow\ \therefore\ Tan^2 x = Sec^2 x - 1$ or $Sec^2 x - Tan^2 x = 1$

(3) $Sin2x = 2Sin\,x\,Cos\,x$ (4) $Tan2x = \frac{2Tan\,x}{1 - Tan^2 x}$

(5) $Cos2x = Cos^2 x - Sin^2 x = 2Cos^2 x - 1 = 1 - 2Sin^2 x$

As a result of that:

$$\boxed{\begin{aligned} Sin^2 x &= \frac{1}{2} - \frac{1}{2}Cos2x \\ Cos^2 x &= \frac{1}{2} + \frac{1}{2}Cos2x \end{aligned}}$$

Examples

Example (1)

Find : $\int \left(Sin^2 2x + Cos^2 2x\right) dx$

Answer

$\because\ Sin^2 2x + Cos^2 2x = 1 \ \Rightarrow\ \therefore\ \int \left(Sin^2 2x + Cos^2 2x\right) dx = \int dx = x + c$

Example (2)

Find : $\int Sin3x\,Cos3x\ dx$ **Answer** $\int \frac{1}{2}\ Sin\ 6x\ dx \ = \frac{-1}{12}\ Cos\ 6x + c$

Example (3)

Find : $\int Sec^2 \frac{\pi}{6}\, dx$

Find : $\int Sec^2 \frac{x}{6}\, dx$

Answer

$$\int Sec^2 \frac{\pi}{6} dx = x\,Sec^2 \frac{\pi}{6} + c$$

$\left(\text{Notice that } Sec^2 \frac{\pi}{6} \text{ is constant} \right)$

$$\int Sec^2 \frac{x}{6}\, dx = 6\,Tan \frac{x}{6} + c$$

Example (4)

Find : $\int (Sin\ x + Cos\,x)^2\, dx$

Answer

$\because (Sin\ x + Cos\,x)^2 = Sin\ ^2 x + Cos\ ^2 x + 2Sin\ x\,Cos\,x \ \Rightarrow$ but $\because\ Sin\ ^2 x + Cos\ ^2 x = 1$

And $\because\ 2Sin\ x\,Cos\,x = Sin\,2x \Rightarrow \therefore \int (Sin\ x + Cos\,x)^2\, dx = \int (1 + Sin\ 2x) dx = x - \frac{1}{2} Cos\,2x + c$

Example (5)

Find : $\int (1 - Cos\ x)^2\, dx$

Answer

$\because\ 1 - 2Cos\,x + Cos^2 x = 1 - 2Cos\,x + \frac{1}{2} + \frac{1}{2} Cos\,2x$

$\therefore \int (1 - Cos\ x)^2\, dx = \int \left(\frac{3}{2} - 2Cos\,x + \frac{1}{2} Cos\,2x \right) dx = \frac{3}{2} x - 2Sin\ x + \frac{1}{4} Sin\,2x + c$

Example (6)

Find : $\int (1 + Sin\,x)^2 dx$

Answer

$1 + 2Sin\,x + Sin\ ^2 x = 1 + 2Sin\,x + \frac{1}{2} - \frac{1}{2} Cos\,2x = \frac{3}{2} + 2Sin\,x - \frac{1}{2} Cos\,2x$

$\therefore \int (1 + Sin\,x)^2\, dx = \int \left(\frac{3}{2} + 2Sin\,x - \frac{1}{2} Cos\,2x \right) dx = \frac{3}{2} x - 2Cos\,x - \frac{1}{4} Sin\,2x + c$

This year, we will discuss three new kinds of Integration

Integration of the reciprocal trigonometric functions

Integration of the Exponential functions

Integration of the Logarithmic functions

1 LESSON

The Integration of the reciprocal of trigonometric functions

Rules

Integration
$\int Sec\,x\,Tan\,x\,dx = Sec\,x + c$
$\int Cosec\,x\,Cot\,x\,dx = -Cosec\,x + c$
$\int Cosec^2\,x\,dx = -Cot\,x + c$
$\int Sec(a\,x+b)\,Tan(a\,x+b)\,dx = \frac{Sec(a\,x+b)}{a} + c$
$\int Cosec(a\,x+b)\,Cot(a\,x+b)\,dx = \frac{-Cosec(a\,x+b)}{a} + c$
$\int Cosec^2(a\,x+b)\,dx = \frac{-Cot(a\,x+b)}{a} + c$

Examples

Example (1)

$$Find: \int \frac{1}{1-Cos^2x}dx$$

Answer

$$\int \frac{1}{1-Cos^2x}\,dx = \int \frac{1}{Sin^2x}\,dx = \int Cosec^2x\,dx = -Cot\,x + c$$

Notes

$\because\ Sin^2x + Cos^2x = 1$

$\therefore\ Sin^2x = 1 - Cos^2x$

Example (2)

$$Find: \int \frac{Cos\,x}{1-Cos^2x}dx$$

Answer

$$\int \frac{Cos\,x}{1-Cos^2x}\,dx = \int \frac{Cos\,x}{Sin^2x}\,dx = \int \frac{Cos\,x}{Sin\,x} \times \frac{1}{Sin\,x}\,dx = \int Cosec\,x\,Cot\,x\,dx = -Cosec\,x + c$$

Example (3)

$$Find: \int Cosec^2\left(\frac{x+3}{2}\right)dx$$

Answer

$$\int Cosec^2\left(\frac{1}{2}x+\frac{3}{2}\right)dx = \frac{-Cot\left(\frac{1}{2}x+\frac{3}{2}\right)}{\frac{1}{2}} + c = -2Cot\left(\frac{1}{2}x+\frac{3}{2}\right) + c$$

Example (4)

Find : $\int 1+Cot^2(3x-1)dx$

Answer

$$\int Cosec^2(3x-1)dx=\frac{-Cot(3x-1)}{3}+c=-\frac{1}{3}Cot(3x-1)+c$$

Notes

$\because 1+Cot^2x=Cosec^2x$

Example (5)

Find : $\int 6\ Sec\,2x\left(Tan\,2x+Cos^2\,2x\right)\,dx$

Answer

....

....

....

....

....

....

.... $\boxed{Ans:\ 3Sec\,2x+3Sin\,2x+c}$

Example (6)

Find : $\int\frac{Sin^2(2x-3)}{1-Cos(2x-3)}dx$

Answer

$$\int\frac{1-Cos^2(2x-3)}{1-Cos(2x-3)}\,dx=\int\frac{\cancel{[1-Cos(2x-3)]}[1+Cos(2x-3)]}{\cancel{[1-Cos(2x-3)]}}\,dx=\int 1+Cos(2x-3)\,dx$$

$$=x+\frac{Sin(2x-3)}{2}+c$$

Example (7)

Find : $\int Tan^2x+2Sin^2x\ \ dx$

Answer

$$\int(Sec^2x-1)+2\left(\frac{1}{2}-\frac{1}{2}Cos\,2x\right)\,dx=\int Sec^2x-1+1-Cos\,2x\ \ dx=\int Sec^2x-Cos\,2x\ \ dx$$

$$=Tan\,x-\frac{1}{2}Sin\,2x+c$$

2 LESSON

Integration forms

Alegbraic functions	Exponential functions	Logarithmic functions
Rules There is no special rule for this type . you have to use your experience in algebra by solving : Factorization Difference between two square , difference and sum between two cubes . Conjugates and so on .	**Rules** $(1)\int e^{x}\ dx = e^{x} + c$ $(2)\int e^{kx}\ dx = \frac{e^{kx}}{k} + c$ $(3)\int a^{x}\ dx = \frac{a^{x}}{\ln a} + c$ **Multiplication rule of e^{x}** $\boxed{\int e^{f(x)} f'(x)\ dx = e^{f(x)} + c}$	**Rules** $\boxed{\int \frac{1}{x}\ dx = \ln\lvert x\rvert + c}$ **Division rule of ln** $\boxed{\int \frac{f'(x)}{f(x)}\ dx = \ln\lvert f(x)\rvert + c}$

Examples

Find the integration of the following integrals

Example (1)

$$\int \frac{x^5 - 3x^3 + 2}{x^2}$$

Answer

By using algebra: $\because \int \frac{x^5 - 3x^3 + 2}{x^2}\ dx = \int \frac{x^5}{x^2} - \frac{3x^3}{x^2} + \frac{2}{x^2}\ dx$

$$\therefore \int \left(x^3 - 3x + 2x^{-2}\right)\ dx = \frac{1}{4}x^4 - \frac{3}{2}x^2 - 2x^{-1} + c$$

Example (2)

$$\int \frac{16x^4 - 81}{2x+3}\ dx$$

Answer

By using algebra:

$$\int \frac{\left(4x^2+9\right)\left(2x-3\right)\left(2x+3\right)}{\left(2x+3\right)}\ dx = \int \left(4x^2+9\right)\left(2x-3\right)\ dx = \int \left(8x^3 - 12x^2 + 18x - 27\right)\ dx$$

$$= \frac{8x^4}{4} - \frac{12x^3}{3} + \frac{18x^2}{2} - 27x = 2x^4 - 4x^3 + 9x^2 - 27x + c$$

Example (3)

$$\int \frac{8x^3+27}{2x+3}\,dx$$

Answer

By using algebra :

$$\int \frac{(2x+3)\left(4x^2-6x+9\right)}{2x+3}\,dx = \int\left(4x^2-6x+9\right)dx = \frac{4}{3}x^3-3x^2+9x+c$$

Example (4)

$$\int \frac{x-4}{\sqrt{x}+2}\,dx$$

Answer

By using algebra :

$$\int \frac{\left(\sqrt{x}-2\right)\left(\sqrt{x}+2\right)}{\left(\sqrt{x}+2\right)}\,dx = \int\left(\sqrt{x}-2\right)\,dx = \frac{2}{3}x^{\frac{3}{2}}-2x+c$$

Example (5)

$$\int x^5\left(1-\frac{1}{x}\right)^5\,dx$$

Answer

By using algebra :

$$\int\left[x\left(1-\frac{1}{x}\right)\right]^5 dx = (x-1)^5\,dx = \frac{1}{6}(x-1)^6+c$$

$(6)\ \int e^{7x}\,dx$ **Answer** $\frac{e^{7x}}{7}+c$	$(7)\ \int 3\sqrt{2}\ e^{-\sqrt{2}n}\,dn$ **Answer** $\frac{3\sqrt{2}\ e^{-\sqrt{2}n}}{-\sqrt{2}}+c=-3\ e^{-\sqrt{2}n}+c$	$(8)\ \int 8e^{\frac{-3}{4}y}\,dy$ **Answer** $\frac{8e^{\frac{-3}{4}y}}{\frac{-3}{4}}+c=\frac{-32e^{\frac{-3}{4}y}}{3}+c$
$(9)\ \int \frac{2}{x}\,dx$ **Answer** $2\int \frac{\overset{f'(x)}{1}}{\underset{f(x)}{x}}\,dx = 2\ln\lvert x\rvert+c$	$(10)\ \int \frac{7}{x\ln 3}\,dx$ **Answer** $\frac{7}{\ln 3}\int \frac{\overset{f'(x)}{1}}{\underset{f(x)}{x}}\,dx=\frac{7}{\ln 3}\ln\lvert x\rvert+c$	$(11)\ \int x^{2e}+e^{3x}\,dx$ **Answer** $\frac{x^{2e+1}}{2e+1}+\frac{e^{3x}}{3}+c$

Example (12)

$$\int \frac{\ln x^2}{x \ln x^3}\, dx$$

Answer

By using algebra : $\int \frac{2 \cancel{\ln x}}{3x \cancel{\ln x}}\, dx = \int \frac{2}{3x}\, dx \xrightarrow{\text{By using ln rule}} \frac{2}{3}\int \frac{1}{x}\, dx = \frac{2}{3}\ln|x| + c$

Example (13)

$$\int \frac{3e^{x} - 2e^{2x}}{2e^{x}}\, dx$$

Answer

$$\frac{1}{2}\int \frac{3e^{x}}{e^{x}} - \frac{2e^{2x}}{e^{x}}\, dx = \frac{1}{2}\int 3 - 2e^{x}\, dx = \frac{1}{2}\left[3x - 2e^{x}\right] + c = \frac{3}{2}x - e^{x} + c$$

Example (14)

$$\int e^{\operatorname{Tan} x}\, \operatorname{Sec}^2 x\, dx$$

Answer

$f(x)$

$$\int e^{\operatorname{Tan} x}\, \operatorname{Sec}^2 x\, dx = e^{\operatorname{Tan} x} + c$$

$f'(x)$

Example (15)

$$\int \operatorname{Sin} x\, e^{\operatorname{Cos} x}\, dx$$

Answer

$f(x)$

$$-\int -\operatorname{Sin} x\, e^{\operatorname{Cos} x}\, dx = -e^{\operatorname{Cos} x} + c$$

$f'(x)$

Example (16)

$$\int 4x\, e^{x^2+1}\, dx$$

Answer

$f(x)$

$$2\int 2x\, e^{x^2+1}\, dx = 2e^{x^2+1} + c$$

$f'(x)$

Example (17)

$\int (x-3)\, e^{x^2-6x+6}\ dx$

Answer

....

....

.... $\boxed{Ans : \frac{1}{2}\, e^{x^2-6x+6} + c}$

Example (18)

$\int Tan\, x\ dx$

Answer

$$\int \frac{Sin\, x}{Cos\, x}\, dx = -\int \frac{\overset{f(x)}{-Sin\, x}}{\underset{f'(x)}{Cos\, x}}\, dx = -ln\left|Cos\, x\right| + c = ln\left|(Cos\, x)^{-1}\right| + c = ln\left|Sec\, x\right| + c$$

Example (19)

$\int \frac{(3x-1)^2}{3x}\, dx$

Answer

By using algebra : $\int \frac{9x^2-6x+1}{3x}\, dx = \int \frac{9x^2}{3x} - \frac{6x}{3x} + \frac{1}{3x}\, dx = \int 3x - 2 + \frac{1}{3}\left(\frac{1}{x}\right) dx$

$$= \frac{3}{2}x^2 - 2x + \frac{1}{3} ln|x| + c$$

Example (20)

$\int \frac{6x^2+12}{x^3+6x+1}\, dx$

Answer

$$6\int \frac{x^2+2}{x^3+6x+1}\, dx = \frac{6}{3}\int \frac{3x^2+6}{x^3+6x+1}\, dx = 2ln\left|x^3+6x+1\right| + c$$

Example (21)

$\int \frac{2e^x}{e^x+1}\, dx$

Answer

$$2\int \frac{\overset{f'(x)}{e^x}}{\underset{f(x)}{e^x+1}}\, dx = 2ln\left|e^x+1\right| + c$$

Example (22)

$$\int \frac{dx}{4x-1}$$

Answer

$$\frac{1}{4}\int \frac{\overset{f'(x)}{4}}{\underset{f(x)}{4x-1}}dx=\frac{1}{4}\ln\left|4x-1\right|+c$$

Example (23)

$$\int \frac{Sec\,x\,Tan\,x}{Sec\,x-1}\,dx$$

Answer

$$\int \frac{\overset{f'(x)}{Sec\,x\,Tan\,x}}{\underset{f(x)}{Sec\,x-1}}=\ln\left|Sec\,x-1\right|+c$$

Example (24)

$$\int \frac{1}{x\ln x^{2}}\,dx$$

Answer

$$\int \frac{1}{2x\ln x}dx=\frac{1}{2}\int \frac{1}{x\ln x}dx=\frac{1}{2}\int \frac{\overset{f'(x)}{\frac{1}{x}}}{\underset{f(x)}{\ln x}}dx=\frac{1}{2}\ln\left|\ln x\right|+c$$

Example (25)

$$\int \frac{4}{x\ln 3x}\,dx$$

Answer

$$4\int \frac{\overset{f'(x)}{\frac{1}{x}}}{\underset{f(x)}{\ln 3x}}dx=4\ln\left|\ln 3x\right|+c$$

1 LESSON

Techniques of integration

Introduction

Integration is more chanllenging than differentiation, in finding the derivative of a function it is obvious which differential formula should apply. But it may not be obvious which technique we should use to integrate a given function .

So in order to find the integral of the **product or quotient** between two functions :

Ask yourself **Can the integral be solved directly by the previous methods like :**

Algebraic method | **Trigonometric rules** | **Exponential method** | **logarithmic method**

If Not, use :

1st Technique

The U - Substitution

The idea behind using the substitution method is to replace a relatively complicated integral by a simpler one,this is accomplished by changing the original variable x into u or z or ....

Steps

(1) Assume that a complicated function $= u$

(2) Integrate the function by any of the previous rules .

(3) If your function becomes more complicated than the original function $\Rightarrow$ " change your assumption "

(4) Return the function back to its original variable .

(5) Simplify your integral to the simplest form .

Examples

Example (1)

Find : $\int 2e^{x}\left(e^{x}+1\right)^{2}\ dx$

Ask your self

(1) Integration by algebra or rules ×

(2) Integration by Substitution ✓

Answer

Step (1) : Let $u = e^{x}+1 \Rightarrow du = e^{x}\ dx \rightarrow dx = \dfrac{du}{e^{x}}$

$$\therefore\ I = \int \frac{2\,\cancel{e^{x}}\,u^{2}}{\cancel{e^{x}}}du = \int 2u^{2}\ du$$

As you see the integration here is more simpler than the original one

Step (2) : $I = 2\times\dfrac{u^{3}}{3}+c = \dfrac{2}{3}u^{3}+c \Rightarrow$ Step (3) : $I = \dfrac{2}{3}\left(e^{x}+1\right)^{3}+c$