Mixed examples

Example (6)

Determine the local maximum and minimum points of $f(x)=\begin{cases} x^3+3x^2 & , x\le 0 \\ x^2-2x & , x>0 \end{cases}$, then find the intervals of the convexity up and down, and get the inflection point and equation of tangent .

Answer

$\because f(x)$ is continous at $x=0$ as $f(0)=f(0^-)=f(0^+)=0$

(i) Check if : $f'(0^+)\ne f'(0^-)$ by using defferentiability (ii) $f'(x)=0$

(i) For $f'(0^-)$: $\lim\limits_{h\to 0^+}\dfrac{f(0+h)-f(0)}{h}\Rightarrow \lim\limits_{h\to 0^+}\dfrac{h^3+3h^2-0}{h}\Rightarrow \lim\limits_{h\to 0^+}\dfrac{h(h^2+3h)}{h}=0$

For $f'(0^+)$: $\lim\limits_{h\to 0^+}\dfrac{f(0+h)-f(0)}{h}\Rightarrow \lim\limits_{h\to 0^+}\dfrac{h^2-2h-0}{h}\Rightarrow \lim\limits_{h\to 0^+}\dfrac{h(h-2)}{h}=-2$

$\boxed{\therefore\ f'(0^-)\ne f'(0^+)}$

For Critical point

$\boxed{\therefore x=0}$ is a critical point

$\because\ f'(x)=\begin{cases} 3x^2+6x & , x\le 0 \\ 2x-2 & , x>0 \end{cases}$

For $x\le 0$: $3x^2+6x=0 \Rightarrow 3x(x+2)=0$

$\boxed{x=0}$ or $\boxed{x=-2}$

For $x>0$: $2x-2=0 \Rightarrow \boxed{x=1}$

$\therefore$ The critical points are:

$\boxed{x=0}$ and $\boxed{x=-2}$ and $\boxed{x=1}$

$\because\ f''(x)=\begin{cases} 6x+6 & , x\le 0 \\ 2 & , x>0 \end{cases}$

$\therefore\ f''(-2)=6(-2)+6=-6<0$ (L. maximum)

$f''(0)=6(0)+6=6>0$ (L. minimum)

$f''(1)=2>0$ (L. minimum)

Then substitute in the original function

$\therefore\ (-2,4)$ is a local maximum point

$\therefore\ (0,0)$ is a local minimum point

$\therefore\ (1,-1)$ is a local minimum point

For Inflection point

$\boxed{\therefore x=0}$ is not an inflection point

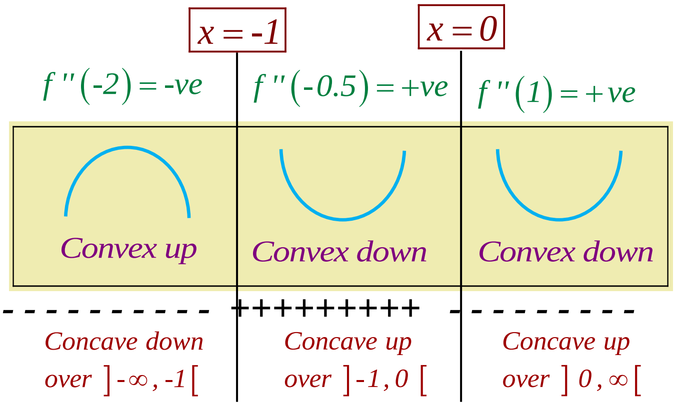
$\therefore\ f''(x)=\begin{cases} 6x+6 & , x\le 0 \\ 2 & , x>0 \end{cases}$

For $x\le 0$: $6x+6=0 \Rightarrow 6(x+1)=0$

$\boxed{x=-1\ (agreed)}$

For $x>0$: $2=0$ Can't be

$\therefore$ The Inflection point is at $x=-1$:



Then the inflection point is $(-1,2)$

Equation of tangent at this point

$\because$ The point of tangency is $(-1,2)$

$\because\ f'(x)=3x^2+6x \Rightarrow \therefore$ at $x=-1 \Rightarrow \boxed{\dfrac{dy}{dx}=-3}$

$\therefore\ y-2=-3(x+1) \Rightarrow \boxed{\therefore y+3x+1=0}$

Example (7)

Determine the absolute maximum and minimum points of $f(x)=x\,|x-4|$ over $[0,4]$. Then find the intervals of the convexity, also find the inflection points and the equation of tangent if exist .

Answer

$$\because\ f(x)=\begin{cases} x(x-4) & , x\geq 4 \\ x(4-x) & , x<4 \end{cases} \Rightarrow \therefore\ f(x)=\begin{cases} x^2-4x & , x\geq 4 \\ 4x-x^2 & , x<4 \end{cases}$$

$\because\ f(x)$ is continous at $x=4$ as $f(4)=f(4^-)=f(4^+)=0$

(i) Check if : $f'(4^+)\neq f'(4^-)$ by using defferentiability $\quad (ii)\ f'(x)=0$

$$(i)\text{For } f'(4^-):\ \lim_{h\to 0^+}\frac{f(4+h)-f(4)}{h}\Rightarrow \lim_{h\to 0^+}\frac{4(4+h)-(4+h)^2-\left[4(4)-(4)^2\right]}{h}=4$$

$$\text{For } f'(4^+):\ \lim_{h\to 0^+}\frac{f(4+h)-f(4)}{h}\Rightarrow \lim_{h\to 0^+}\frac{(4+h)^2-4(4+h)-\left[(4)^2-4(4)\right]}{h}=-4$$

$$\therefore\ f'(4^-)\neq f'(4^+)$$

For Critical point

$\therefore x=4$ is a critical point

$$\because\ f'(x)=\begin{cases} 2x-4 & , x\geq 4 \\ 4-2x & , x<4 \end{cases}$$

For $x\geq 4$: $2x-4=0 \Rightarrow 2x=4$

$x=2$ (refused)

For $x<4$: $4-2x=0 \Rightarrow x=2$ (agreed)

$\therefore$ The critical point is: $x=2$ and $4\in[0,4]$

$\therefore\ f(0)=0 \Rightarrow$ point is $(0,0)$

$f(2)=4 \Rightarrow$ point is $(2,4)$

$f(4)=0 \Rightarrow$ point is $(4,0)$

Then substitute in the original function

$\therefore\ (2,4)$ is an absolute maximum point

$(0,0)\,,(4,0)$ are absolute minimum point

For Inflection point

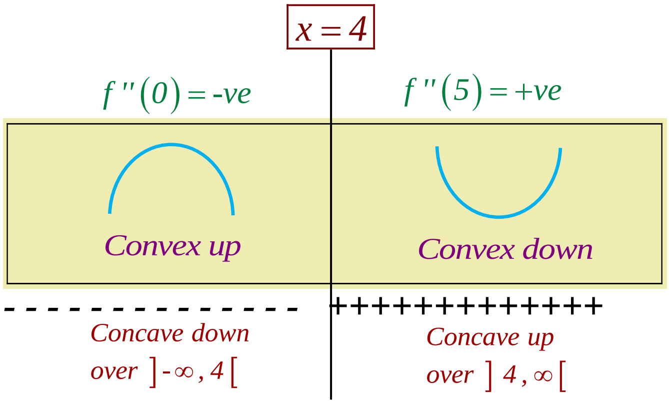
$\therefore x=4$ is not an inflection point

$$\therefore\ f''(x)=\begin{cases} 2 & , x\geq 4 \\ -2 & , x<4 \end{cases}$$

For $x\geq 4$: $2=0$ Can't be

For $x<4$: $-2=0$ Can't be

$\therefore$ There is no Inflection points



$\because$ There is no inflection point

$\therefore$ There is no tangent

Example (8)

Determine the intervals of concavity of the function $f(x)=\dfrac{x}{1-x^2}$, then prove that the measure of the tangent angle at the inflection point of the curve is $\dfrac{\pi}{4}$

Answer

The domain of $f(x)$ is $R-\{\pm 1\}$ $\Rightarrow$ $f(x)$ is continous on $R-\{\pm 1\}$

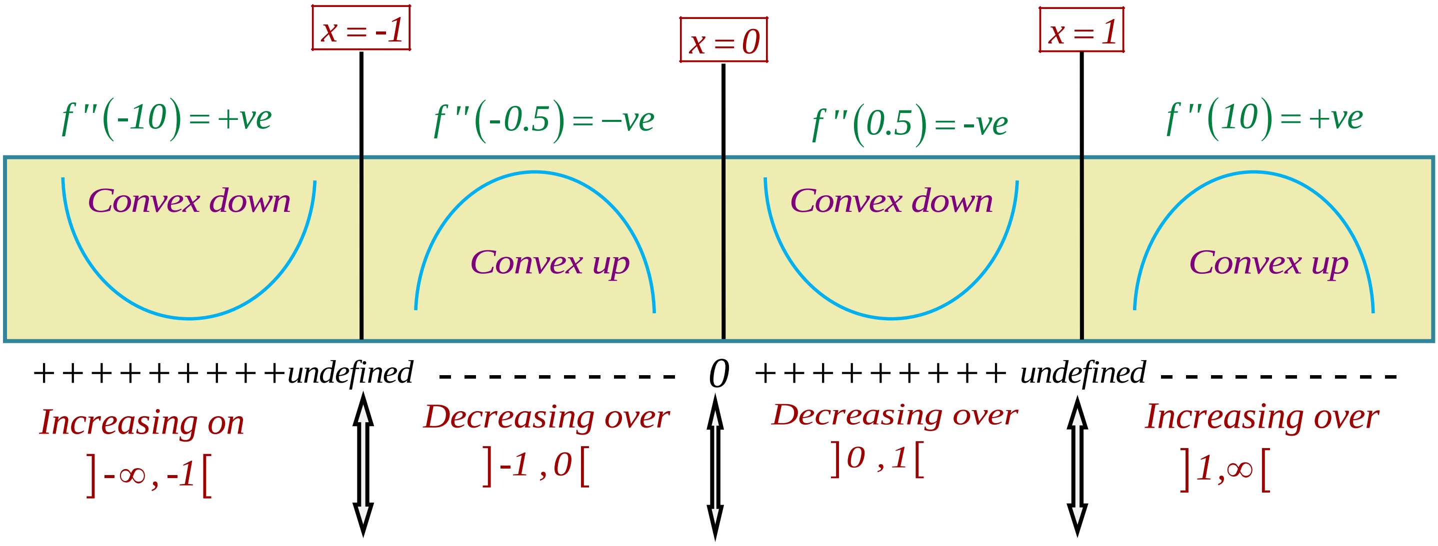
$$f'(x)=\frac{(1-x^2)-x(-2x)}{(1-x^2)^2}=\frac{x^2+1}{(1-x^2)^2}$$

$$f''(x)=\frac{2x(1-x^2)^2-4x(x^2+1)(1-x^2)}{(1-x^2)^4}=\frac{2x(1-x^2)\left[1-x^2+2x^2+2\right]}{(1-x^2)^4}$$

$$f''(x)=\frac{2x\left[x^2+3\right]}{(1-x^2)^3}\Rightarrow \text{ when } f''(x)=0$$

$(1)\ \therefore\ 2x\left[x^2+3\right]=0 \Rightarrow \boxed{\therefore x=0}$ or $\boxed{x^2=-3\ (\text{can't be})}$

$(2)\ f''(x)$ is undefined when $(1-x^2)^3=0 \Rightarrow \therefore\ x^2=1 \Rightarrow \boxed{\therefore\ x=\pm 1\ (\text{refused})}$



Then the inflection point is at $x=0$ is $(0\ ,0)$

Equation of tangent at this point

$\because$ The point of tangency is $(0\ ,0)$

$\because\ f'(x)=\dfrac{x^2+1}{(1-x^2)^2} \Rightarrow \therefore$ at $x=0 \Rightarrow \boxed{\therefore\ Tan\ \theta=\dfrac{dy}{dx}=1} \Rightarrow \boxed{\therefore\ \theta=Tan^{-1}1=\dfrac{\pi}{4}}$

Example (9)

Find a , b , c and d in the function : $f(x) = ax^3 + bx^2 + cx + d$ such that $f(x)$:

(1) Passes through $(1,2)$ (2) has an inflection point on $(2,1)$

(3) The tangent to it at its inflection point is horizontal .

Answer

As the curve passes through $(1,2)$

$\therefore$ At $x = 1$ and $f(x) = 2 \Rightarrow$ then $\boxed{a + b + c + d = 2 --- (1)}$

For the inflection point $(2,1)$:

Remember that:

$\boxed{x = 2 \text{ came from the second derivative } [f''(x) = 0] \text{ and was substituted in the original } f(x)}$

So $f(2) = 8a + 4b + 2c + d \Rightarrow \boxed{8a + 4b + 2c + d = 1 --- (2)}$

To get the second derivative: $f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

At $\boxed{x = 2}$: $f''(2) = 12a + 2b \Rightarrow \boxed{6a + b = 0 --- (3)}$

The tangent of the curve to the inflection point is horizontal means that $f'(x) = 0$ at $x = 2$

$f'(x) = 3ax^2 + 2bx + c \Rightarrow f'(2) = 12a + 4b + c \Rightarrow \boxed{12a + 4b + c = 0 --- (4)}$

Subtract $(2) - (1)$: $\boxed{7a + 3b + c = -1 --- (5)}$

Subtract $(4) - (5)$: $\boxed{5a + b = 1 --- (6)}$

Subtract $(3) - (6)$: $\boxed{a = -1}$

Substitute in (6): $\boxed{b = 6} \Rightarrow$ substitute in (5) : $\boxed{c = -12} \Rightarrow$ substitute in (2): $\boxed{d = 9}$

Example (10)

find a and b such that $f(x)=x^3+ax^2+bx$ *has two critical points at* $x=-1$ *and* $x=2$, *then find the point of inflection if exists.*

Answer

To get the critical points :

$$f'(x)=3x^2+2ax+b \Rightarrow \text{ when } f'(x)=0 \Rightarrow 3x^2+2ax+b=0$$

$$\text{At } x=-1 \Rightarrow 3-2a+b=0 \Rightarrow \boxed{b-2a=-3---(1)}$$

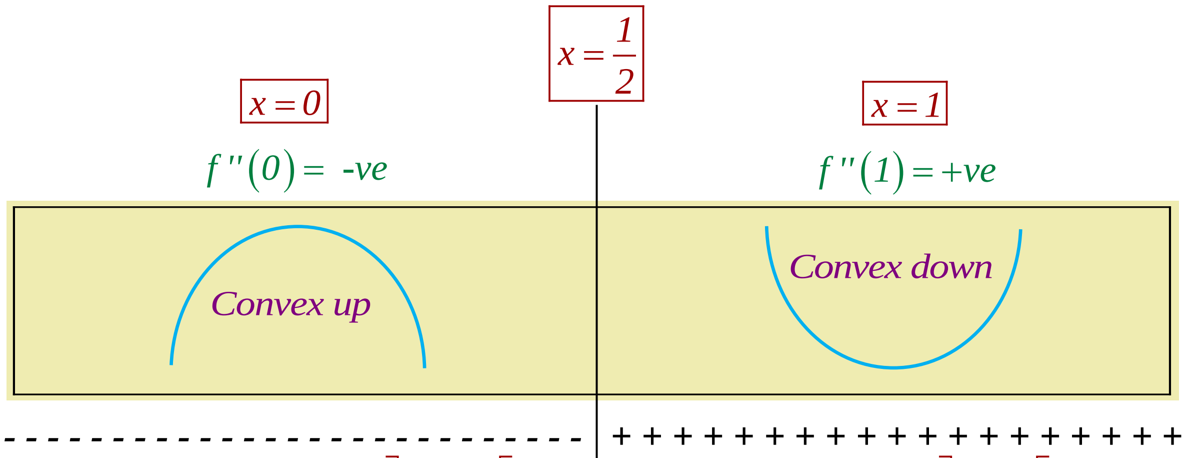
$$\text{At } x=2 \Rightarrow 12+4a+b=0 \Rightarrow \boxed{b+4a=-12---(2)}$$

$$\text{Subtract } (2)-(1): 6a=-9 \Rightarrow \boxed{a=-\frac{9}{6}=-\frac{3}{2}}$$

$$\text{Substitute in } (1): \boxed{b=-6} \text{ then } f(x)=x^3-\frac{3}{2}x^2-6x$$

To get the inflection point : $f'(x)=3x^2-3x-6$

$$f''(x)=6x-3 \Rightarrow f''(x)=0 \Rightarrow 6x-3=0 \Rightarrow \boxed{\therefore\ x=\frac{1}{2}}$$



Concave down over $\left]-\infty,\frac{1}{2}\right[$ Concave up over $\left]\frac{1}{2},\infty\right[$

So, since there is a concave down then concave up, that means that $x=2$ *is an inflection point*

then substitute in the original function: $f\left(\frac{1}{2}\right)=\left(\frac{1}{2}\right)^3-\frac{3}{2}\left(\frac{1}{2}\right)^2-6\left(\frac{1}{2}\right)=-3\frac{1}{4}$

Then the inflection point is $\left(\frac{1}{2},-\frac{13}{4}\right)$

Example (11)

Find the equation of the second degree if the function passes through $(0,0)$ and $(2,-12)$ and the point $(2, f(2))$ is a critical point, and find its concavity.

Answer

The equation of the second degree: $f(x) = ax^2 + bx + c$

At point $(0,0)$: $\boxed{c=0} \Rightarrow f(x) = ax^2 + bx$

At point $(2,-12)$: $\Rightarrow -12 = a(2)^2 + b(2) \Rightarrow 4a + 2b = -12 \quad (\div 2)$

$$\boxed{2a + b = -6 ---(1)}$$

When the critical point is $(2, f(2))$:

$f'(x) = 2ax + b \Rightarrow f'(2) = 4a + b \Rightarrow \boxed{4a + b = 0 ---(2)}$

Subtract $(2)-(1)$: $2a = 6 \Rightarrow \boxed{a=3}$ substitute in (2): $\boxed{b=-12}$

Then the function is $f(x) = 3x^2 - 12x$

To discuss concavity: $f'(x) = 6x - 12 \Rightarrow f''(x) = 6 > 0 \Rightarrow$ concave up (convex down)

Example (12)

Find the equation of the third degree if the curve passes through $(1,0)$, the point $(2,1)$ is an inflection point and the equation of the tangent at $(2,1)$ is $3x + 2y = 8$

Answer

The equation of the third degree: $f(x) = ax^3 + bx^2 + cx + d$

At point $(1,0)$: $\Rightarrow \boxed{\therefore a + b + c + d = 0 ---(1)}$

$\because$ Point $(2,1)$ is the inflection point: $\Rightarrow \boxed{\therefore 1 = 8a + 4b + 2c + d ---(2)}$

$\therefore f'(x) = 3ax^2 + 2bx + c \Rightarrow f''(x) = 6ax + 2b \Rightarrow f''(2) = 0$

$\therefore 12a + 2b = 0 \Rightarrow \boxed{\therefore 6a + b = 0 ---(3)}$

The equation of the tangent passing through $(2, 1)$: $3x + 2y = 8$ means that :

The slope of the tangent is $\frac{-3}{2}$ at $x = 2 \Rightarrow \therefore f'(2) = \frac{-3}{2}$

So $\because f'(x) = 3ax^2 + 2bx + c \Rightarrow \boxed{\therefore 12a + 4b + c = \frac{-3}{2} ---(4)}$

Subtract $(2)-(1)$: $\boxed{7a + 3b + c = 1 ---(5)}$

Subtract $(4)-(5)$: $\boxed{5a + b = -2.5 ---(6)}$

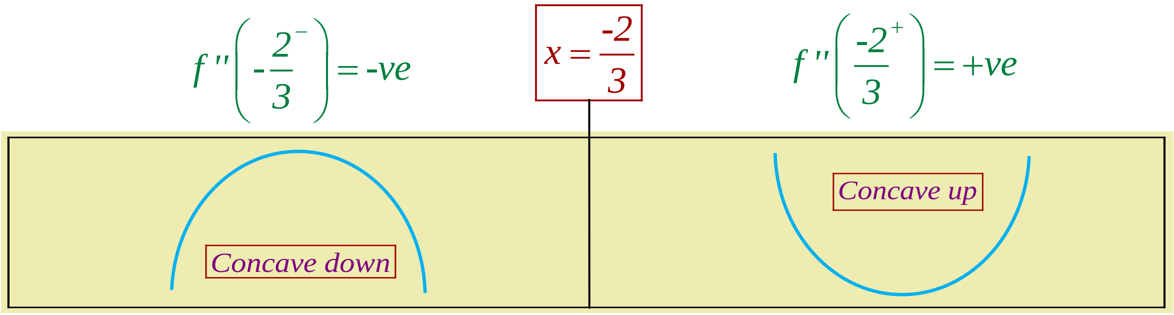
Subtract $(3)-(6)$: $\boxed{a = 2.5}$ and $\boxed{b = -15}$ and $\boxed{c = \frac{57}{2}}$ and $\boxed{d = -16}$

Example (13)

If $y = f(x)$ represents a function of a third degree polynomial, such that $f''(x) < 0$ when $x < \frac{-2}{3}$; $f''(x) > 0$ when $x > \frac{-2}{3}$ and the curve of the function passes through the point $(1,6)$ and there exist a critical point at $(-1,2)$. Find the equation of this curve and the type of each critical point.

Answer

Let the equation be : $f(x) = a x^3 + b x^2 + c x + d$



Since there is a concave down then cancave up , that means that $x = \frac{-2}{3}$ is an inflection point

When $f''(x) = 0$, so $f'(x) = 3ax^2 + 2bx + c \Rightarrow f''(x) = 6ax + 2b \Rightarrow 6ax + 2b = 0$

Then put $\boxed{x = \frac{-2}{3}} \Rightarrow -4a + 2b = 0 \quad (\div -2) \Rightarrow \therefore 2a - b = 0 \rightarrow \boxed{b = 2a ---(1)}$

The curve of the function passes through $(1,6)$: $\boxed{a + b + c + d = 6 ---(2)}$

The function has critical point at $(-1,2)$: $\because f'(x) = 3a x^2 + 2bx + c$

Critical point exists when $f'(x) = 0 \rightarrow \therefore 3a x^2 + 2bx + c = 0$

At $x = -1$: $\rightarrow \boxed{3a - 2b + c = 0 ---(3)}$ and at $(-1,2)$: $\boxed{-a + b - c + d = 2 ---(4)}$

From $(2),(4)$: by additional : $2b + 2d = 8 \rightarrow \boxed{b + d = 4 ---(5)}$

Substitute in (2): $a + c + 4 = 6 \rightarrow \boxed{c = 2 - a ---(6)}$

Substitute $(1) \& (6)$ in (3): $3a - 2(2a) + 2 - a = 0$

$\therefore 3a - 4a + 2 - a = 0 \rightarrow -2a = -2 \rightarrow \boxed{a = 1} \rightarrow$ substitute in (1) : $\boxed{b = 2}$

Substitute in (6): $\boxed{c = 2 - 1 = 1}$

Substitute in (5): $2 + d = 4 \rightarrow \boxed{d = 2} \Rightarrow \boxed{\therefore f(x) = x^3 + 2x^2 + x + 2}$

To find the type of the critical point: $f'(x) = 3x^2 + 4x + 1$

When $f'(x) = 0 \Rightarrow 3x^2 + 4x + 1 = 0 \rightarrow (3x+1)(x+1) = 0 \Rightarrow \boxed{\therefore x = -\frac{1}{3}}$ or $\boxed{x = -1}$

$\therefore f''(x) = 6x + 4 \Rightarrow f''\left(-\frac{1}{3}\right) = 2 > 0$ $\left(\text{local minimum at } x = -\frac{1}{3}\right)$

$f''(-1) = -2 < 0$ $(\text{local maximun at } x = -1)$

Tracing graph

To sketch the graph of a function , we follow :

(1) Find $f'(x)$, $f''(x)$

(2) Use $f'(x)$ for : determining the intervals over which the function is increasing where $f'(x)>0$ and also the intervals over which the function is decreasing where $f'(x)<0$

(3) Use $f''(x)$ and use it to determine the intervals over which the curve is convex upwards where $f''(x)<0$ and the intervals over which the curve is convex downwards where $f''(x)>0$, also we use $f''(x)=0$ to get the points of inflection if they exist .

(4) Determine some points to help us to sketch the graph say :

$i)$ Points of intersection of $f(x)$ with the $x-$axis by solving $f(x)=0$ and also the points of intersection with the $y-$axis $\rightarrow$ $(0, f(0))$.

$ii)$ Other points by substitution by some values of X and find $f(x)$ in each case .

(5) Arrange all the points in a table and plot the points , then complete drawing the graph .

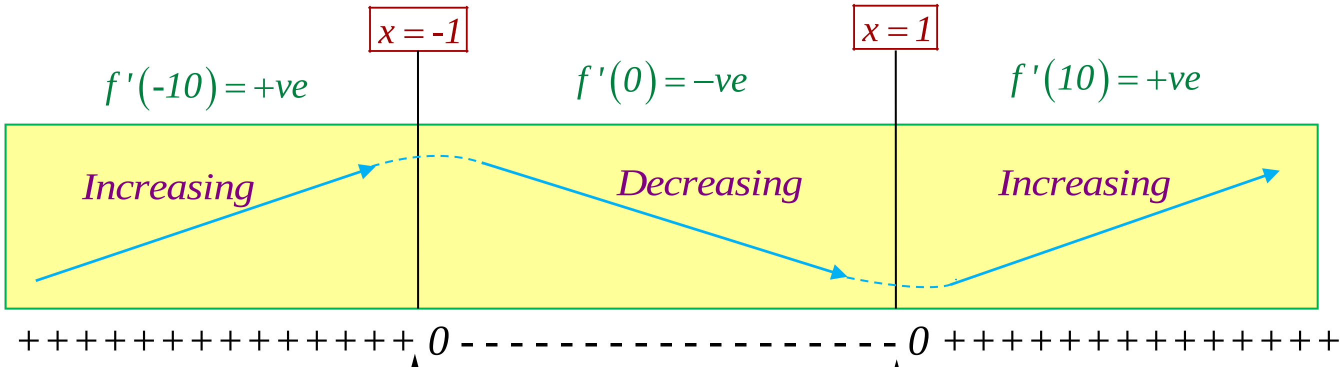
Example (1)

Sketch the graph of the function $f(x)=(x+2)(x-1)^2$

Answer

Step (1): $f(x)=x^3-3x+2 \Rightarrow f'(x)=3x^2-3 \Rightarrow$ for $f'(x)=0 \Rightarrow 3x^2-3=0$

$\therefore\ x^2=1 \Rightarrow \therefore$ $\boxed{x=1}$ or $\boxed{x=-1}$



$f(x)$ is increasing on $]-\infty, -1[$

$f(x)$ is decreasing over $]-1, 1[$

$f(x)$ is increasing over $]1, \infty[$

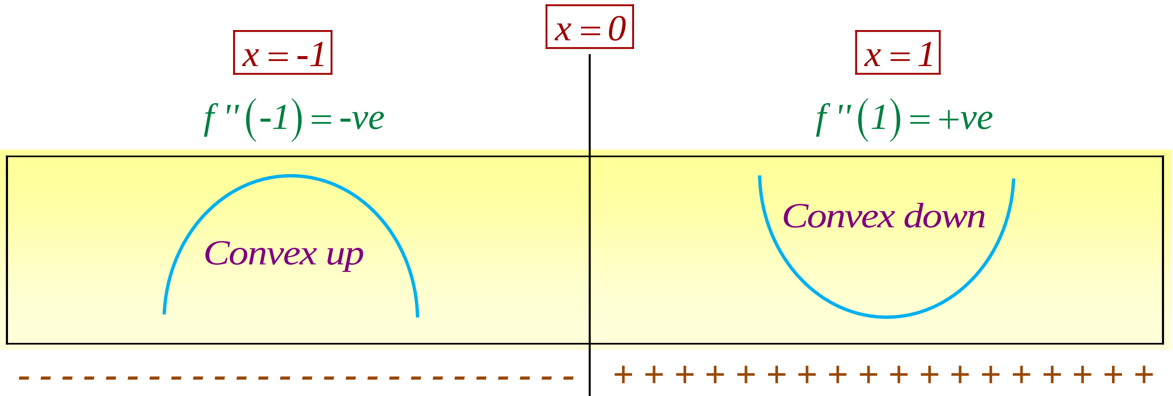
$f(x)$ has a local maximum at $x=-1$

$f(x)$ has a local minimum at $x=1$

So from that : There is a local maximum at $x=-1 \Rightarrow \boxed{f(-1)=4---(1)}$

There is a local minimum at $x=1 \Rightarrow \boxed{f(1)=0---(2)}$

Step (2): $f''(x)=6x$, so when $f''(x)=0 \Rightarrow \therefore\ 6x=0 \Rightarrow \therefore\ x=0$



Concave down over $]-\infty,0[$ Concave up over $]0,\infty[$

So , since there is a **concave down** then **concave up**, this means that $x=0$ is an inflection point , then substitute in the original function : $f(0)=2$

Then the inflection point is $\boxed{f(0)=2 --- (3)}$

So we have made 3 points …… so let us choose another 4 points to be easy to us to draw the graph

Step (3): $\because\ f(x)=x^3-3x+2 \Rightarrow \therefore$ the curve intersects the x – axis in the points

$\boxed{(-2,0),(1,0)}$ and also intersects the y – axis in $\boxed{(0,2)}$

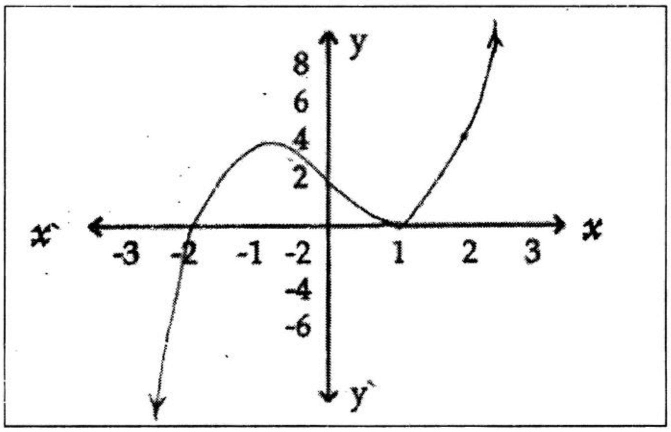
Step (4): Take another points like: $f(3)=20 \Rightarrow f(-3)=-16 \Rightarrow f(2)=4$

Then the curve passes through the points $(3,20)$, $(-3,-16)$ And $(2,4)$

Step (5): arrange all points in the following table:

x	-3	-2	-1	0	1	2	3
y	-16	0	4	2	0	4	20

And the curve of the function is shown in the figure below :



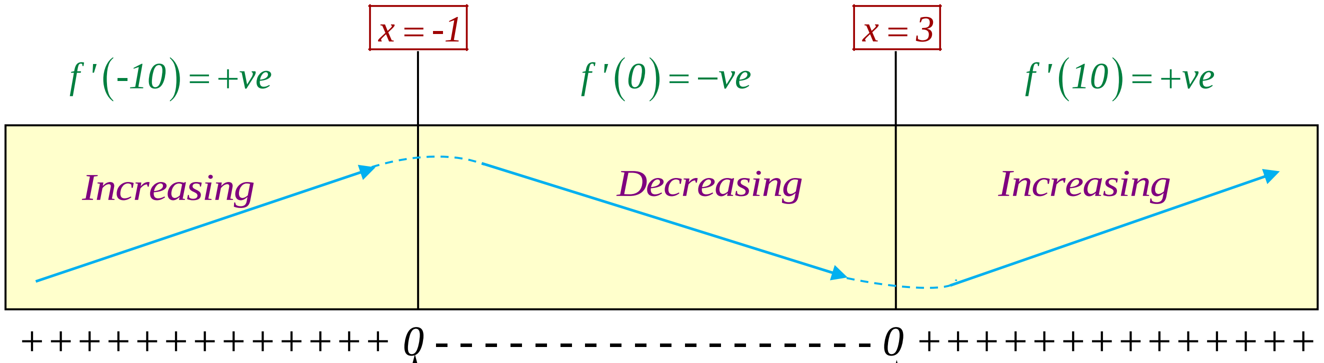
Example (2)

Sketch the graph of the function $f(x)=\frac{1}{3}x^3-x^2-3x+1$

Answer

Step (1): $f'(x)=x^2-2x-3 \Rightarrow$ for $f'(x)=0 \Rightarrow x^2-2x-3=0 \Rightarrow (x-3)(x+1)=0$

$\therefore$ $\boxed{x=3}$ or $\boxed{x=-1}$



$f(x)$ is increasing on $]-\infty,-1[$

$f(x)$ is decreasing over $]-1,3[$

$f(x)$ is increasing over $]3,\infty[$

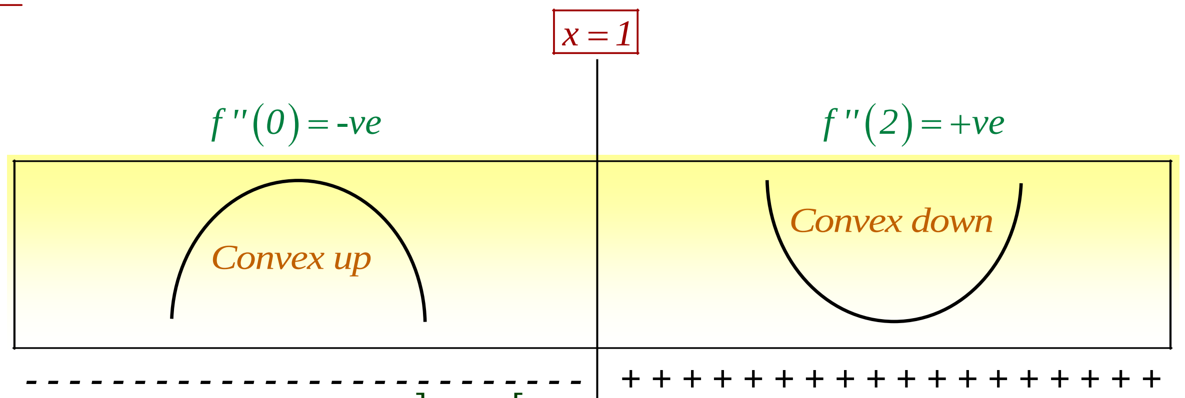
$f(x)$ has a local maximum at $x=-1$

$f(x)$ has a local minimum at $x=3$

So from that : There is a local maximum at $x=-1 \Rightarrow \boxed{f(-1)=\frac{8}{3}---(1)}$

There is a local minimum at $x=1 \Rightarrow \boxed{f(3)=-8---(2)}$

Step (2): $f''(x)=2x-2$, so when $f''(x)=0 \Rightarrow \therefore 2x-2=0 \Rightarrow \therefore x=1$



Concave down over $]-\infty,1[$

Concave up over $]1,\infty[$

So , since there is a **concave down** then **concave up**, this means that $x=1$ is an inflection point , then substitute in the original function : $f(1)=\frac{-8}{3}$

Then the inflection point is $\boxed{f(1)=\frac{-8}{3}---(3)}$

So we have made 3 points …… So let's choose another 4 points to be easy to us to draw the graph

$\underline{Step\ (3)}:\ \because\ f(x)=\frac{1}{3}x^3-x^2-3x+1\ \Rightarrow\ \therefore$ *the curve intersects the* $y-$*axis in the points*

$\boxed{(0,1)}$

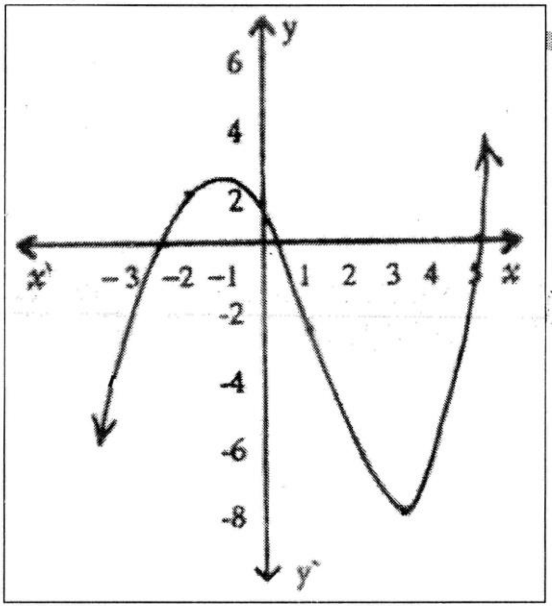
$\underline{Step\ (4)}:$ *Take another points like:* $f(3)=-8\ \Rightarrow\ f(5)=\frac{8}{3}$

Then the curve passes through the points $(3,-8)$, $\left(5,\frac{8}{3}\right)$

$\underline{Step\ (5)}:$ *Arrange all points in the following table:*

x	-3	-1	0	1	-2.6	3
y	-8	$\frac{8}{3}$	1	$-\frac{8}{3}$	0	-8

And the curve of the function is shown in the figure below :



Example (3)

If the curve $y = x^3 + ax^2 + bx$ has an inflection point at $(-1, 11)$:

i) Find the equation of the curve

ii) Find the maximum and minimum local values, then sketch the curve.

Answer

$\because$ $(-1,11)$ lies on the curve : $y = x^3 + ax^2 + bx \Rightarrow \therefore 11 = -1 + a - b \Rightarrow$ $\boxed{\therefore\ a - b = 12 --- (1)}$

And $y' = 3x^2 + 2ax + b \Rightarrow y'' = 6x + 2a$, And $\because$ $(-1,11)$ is an inflection point

$\therefore f''(-1) = 0 \Rightarrow \therefore 0 = -6 + 2a \Rightarrow$ $\boxed{\therefore\ a = 3}$, then from (1): $\boxed{b = -9}$

Then the equation of the curve : $y = x^3 + 3x^2 - 9x$

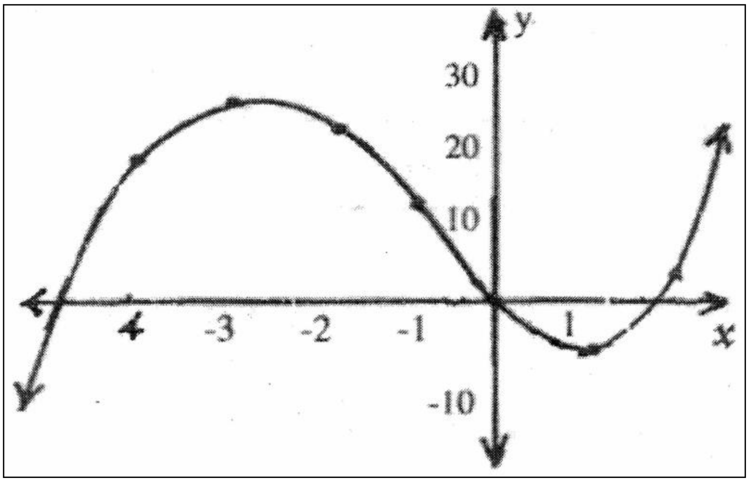
$\therefore y' = 3x^2 + 6x - 9 = 3(x^2 + 2x - 3) = 3(x-1)(x+3)$

So when $y' = 0 \Rightarrow \therefore$ $\boxed{x = 1 \quad Or \quad x = -3}$

Then at $x = 1 \Rightarrow \therefore f''(1) = 12 > 0 \Rightarrow \therefore (1,-5)$ is a local minimum point

And at $x = -3 \Rightarrow \therefore f''(-3) = -12 < 0 \Rightarrow \therefore (-3,27)$ is a local maximum point

x	-4	-3	-2	-1	0	1	2
y	20	27	22	11	0	-5	2



LESSON 2

Solving maxima and minima problems

In most practical life problems which can be expressed as mathematical problems often the aim is to find the great value or less value for a variable such as to find the greatest surface area or the least volume or price ,....etc. $\Rightarrow$ *differentiation gives the best way to solve these problems.*

To solve these problems please follow the following :

(1) *Express the wanted variable* (y) *for which we want to get the maximum or minimum value , as a function in one another variable* (x). *We get this by the given data in the problem .*

(2) *Find the critical points for the function , lying in this domain .*

(3) *Find the value of the function at these critical points and at the ends of the domain , to know the absolute maximum or the absolute minimum for the function .*

Additional rules *beside what we have taken in related time rates*

(1) If $\Delta ABC \sim \Delta DEF \Rightarrow \therefore \frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$

(2) The distance between 2 points (x_1, y_1) and (x_2, y_2) is : $S = \sqrt{(y_2 - y_1)^2 + (x_2 - x_1)^2}$

Example (1)

If the sum of two positive integers is 12 , find the two numbers if :

(i) *Their product is to be maximum* (ii) *The sum of their cubes is to be minimum .*

Answer

Let one of the numbers be $x \Rightarrow \therefore$ the other number $= 12 - x$

(i) Let the product of the two numbers : $y = x(12 - x) \Rightarrow \boxed{\therefore\ y = 12x - x^2 \ ---(1)}$

$\therefore\ y' = 12 - 2x$, so when $y' = 0 \Rightarrow \boxed{x = 6} \Rightarrow$ substitute in $(1) \Rightarrow \boxed{y = 36}$

To check that the product is maximum : Differentiate again

$\therefore\ y'' = -2 < 0$, Then the product of the two numbers are maximum and they are 6 and $12 - 6 = 6$

Thus the maximum value of the product is $y = 36$ at $x = 6$

Then the two real numbers are 6 , 6

(ii) Let the two numbers are $x, 12 - x \Rightarrow$ thus $\boxed{y = x^3 + (12 - x)^3 \ ---(2)}$

$\therefore\ y' = 3x^2 + 3(12 - x)^2(-1) = 3x - 3(144 - 24x + x^2) = 72x - 432$

So when $y' = 0$ at $\boxed{x = 6} \Rightarrow$ this is a critical point

$\therefore$ at $x = 6$: $y = (6)^3 + (6)^3 = 432$

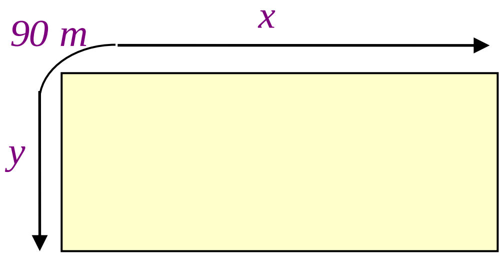
To check that the product is minimum : Differentiate again

$y'' = 72 > 0$, then the sum of the cube of the two numbers are minimum and their values are 6 , 6

Example (2)

A rectangle piece of land is surrounded by a fence of length 180 metres , find the dimensions of the rectangle such that its surface area is to be maximum .

Answer



Let the length be x and the width be y $\Rightarrow$ $2(x+y)=180$

$\therefore\ x+y=90 \Rightarrow \boxed{\therefore\ y=90-x}$

Then the dimensions of the rectangle are x and $90-x$

$\therefore$ The surface area $(S)=x(90-x)=90x-x^2$

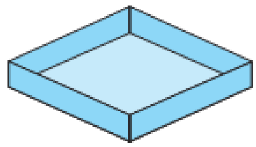
$\therefore\ S'=90-2x \Rightarrow$ when $S'=0 \Rightarrow \therefore\ 2(45-x)=0 \Rightarrow \therefore\ \boxed{x=45}$

And $\because\ S''=-2<0 \Rightarrow \therefore\ S$ is maximum when the two dimensions are 45 , 45 metres .

Example (3)

A rectangular thin metalic lamina has dimensions 10 cm, 16 cm. A square the length of its side is x is cut from each of its four corners then the side of the lamina are turned up to construct a box in the shape of a cuboid whose height is x cm, evaluate x if the volume of the box is to be maximum .

Answer

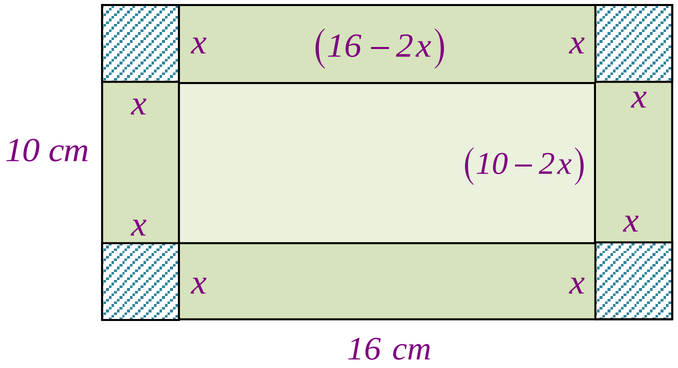


As the square cut from the four corners have sides of length x cm.

$\therefore$ The dimensions of the cuboid are $(16-2x)$, $(10-2x)$ and x

The volume of the cuboid $(V)=L\times W\times H=x(16-2x)(10-2x)$

$\therefore\ V=4x^3-52x^2+160x \Rightarrow (1)$



$\because\ \frac{dv}{dx}=12x^2-104x+160=4(x-2)(3x-20)$

When $\frac{dv}{dx}=0$: $\boxed{\therefore\ x=2}$ or $\boxed{x=\frac{20}{3}}$

$\frac{d^2v}{dx^2}=24x-104$

$\therefore$ At $x=2 \Rightarrow \frac{d^2v}{dx^2}=-56<0$ (agreed)

And at $x=\frac{20}{3} \Rightarrow \frac{d^2v}{dx^2}=56>0$ (refused)

Thus the maximum volume when $x=2$ and its value is $V=4(2)^3-52(2)^2+160(2)=144\ cm^3$

Example (4)

A closed circular cylinder is to be constructed from a fine lamina of area of $24\pi\ cm^2$ Assuming that there is no wasted area , find the dimensions of the cylinder if its volume is to be maximum , and find this volume .

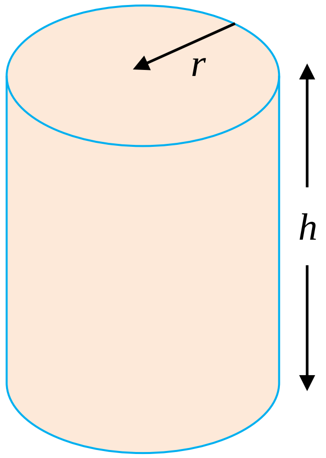
Answer

In this problem, we have to make a relation between Surface area and Volume

Let the radius of the cylinder be r and its height be h $\Rightarrow$ $\because V = \pi r^2 h ---(1)$

But the total surface area $= 2\pi r h + 2\pi r^2 \Rightarrow 24\pi = 2\pi r h + 2\pi r^2 \ (\div 2\pi) \Rightarrow \therefore 12 = r h + r^2$

$\therefore r h = 12 - r^2 \Rightarrow \therefore h = \dfrac{12 - r^2}{r} ---(2)$



Substitute (2) in (1): $\Rightarrow \therefore V = \pi r^2 \times \dfrac{12 - r^2}{r} \Rightarrow \therefore V = 12\pi r - \pi r^3$

$$\therefore \frac{dV}{dr} = 12\pi - 3\pi r^2 = 3\pi(4 - r^2) = 3\pi(2 - r)(2 + r)$$

When $\dfrac{dV}{dr} = 0 \Rightarrow 3\pi(2 - r)(2 + r) = 0 \Rightarrow r = 2$ or $r = -2$ (refused)

$\because \dfrac{d^2V}{dr^2} = -6\pi r \Rightarrow$ when $r = 2$: $\left[\dfrac{d^2V}{dr^2}\right]_{r=2} = -12\pi < 0$

Thus Volume is maximum when $r = 2$, thus the maximum volume of the cylinder is $16\pi\ cm^3$

When $r = 2\ cm$ $\Rightarrow \therefore h = \dfrac{12 - 4}{2} = 4$

Example (5)

The selling price of a bicycle is $Z = 200 - 0.01x$ L.E , where x is the weakly production of these bicycles in a factory. If the factory costs $y = 50x + 2000$ L.E. for producing this number of bicycles, then find the number of bicycles can be produced to give maximum profit . What is the selling price of one bicycle in this case ?

Answer

$\because$ The selling price of one bicycle is $200 - 0.01x$ L.E and the number of bicycles is x .

$\therefore$ The selling price of these bicycles is $x(200 - 0.01x)$ L.E.

$\therefore$ Let the Profit $(P) = Selling - Cost = x(200 - 0.01x) - (50x + 2000) = 150x - 0.01x^2 - 2000$

$\because \dfrac{dP}{dx} = 150 - 0.02x$, so when $\dfrac{dP}{dx} = 0 \Rightarrow 150 - 0.02x = 0 \Rightarrow \therefore x = 7500$

And $\because \dfrac{d^2M}{dx^2} = -0.02 < 0 \Rightarrow \therefore$ the profit is maximum and the selling price of one bicycle is $200 - 0.01 \times 7500 = 125$ L.E.

Example (6)

An open tank is to be made of sheet iron , it must have a square base and sides perpendicular to the base, its capacity is to be 4 m^3 and each 1 m^2 of the sheet iron costs L.E. Find the dimensions of the tank if the cost is minimum .

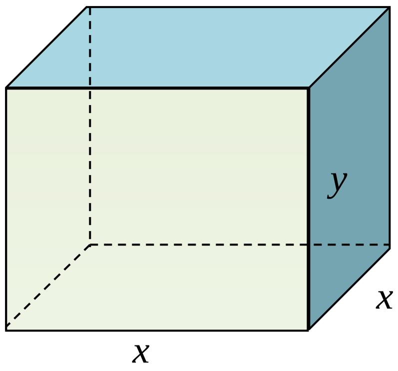
Answer

In this problem, we have to make a relation between surface area and volume

Let x is the length of the side of the square base and let y is the height of the tank .

Its capacity (volume) $= L \times W \times H = 4 \Rightarrow \therefore x^2 y = 4 \Rightarrow \boxed{y = \frac{4}{x^2} ---(1)}$

The surface area of the tank: $\boxed{S = 4xy + x^2 ---(2)}$



By substituting from (1) in (2)

$\therefore S = 4x \times \frac{4}{x^2} + x^2 \Rightarrow \therefore S = \frac{16}{x} + x^2$

$\therefore S' = \frac{-16}{x^2} + 2x \Rightarrow$ when $S' = 0$: then $2x = \frac{16}{x^2}$

$\therefore x^3 = 8 \Rightarrow \boxed{\therefore x = 2}$

$\because S'' = \frac{32}{x^3} + 2 \Rightarrow \therefore$ when $x = 2 \Rightarrow S'' = \frac{32}{8} + 2 = 6 > 0$

$\therefore x = 2 \Rightarrow$ gives a minimum value $\Rightarrow y = \frac{4}{x^2} = \frac{4}{4} = 1$

$\therefore$ The dimensions of the tank are 2 m , 2 m , 1 m

Example (7)

Find the minimum distance between the straight line $x - 2y + 10 = 0$ and the curve $y^2 = 4x$

Answer

The minimum distance between a line and a curve is the perpendicular distance between them

Let a point $(x, y) \in$ to the curve $y^2 = 4x \Rightarrow \boxed{\therefore x = \frac{1}{4} y^2} \longrightarrow$ then the point $(x, y) = \left(\frac{1}{4} y^2, y\right)$

Then the perpendicular distance between the point (x, y) and the line $x - 2y + 10 = 0$ is :

$$S_\perp = \frac{|ax + by + c|}{\sqrt{a^2 + b^2}} \Rightarrow S_\perp = \frac{|x - 2y + 10|}{\sqrt{(1)^2 + (-2)^2}} \Rightarrow S_\perp = \frac{1}{\sqrt{5}}(x - 2y + 10)$$

$$S = \frac{1}{\sqrt{5}}\left(\frac{1}{4} y^2 - 2y + 1\right) \Rightarrow \therefore S' = \frac{1}{2\sqrt{5}} y - \frac{2}{\sqrt{5}}$$

When $S' = 0 \Rightarrow \therefore \frac{1}{2\sqrt{5}} y - \frac{2}{\sqrt{5}} = 0 \Rightarrow \boxed{\therefore y = 4} \Rightarrow \therefore S'' = \frac{1}{2\sqrt{5}} > 0$ [minimum]

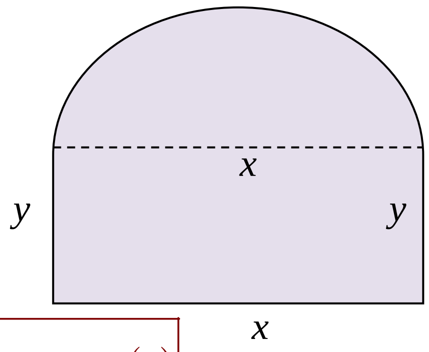
Then from (1): the minimum distance is: $\boxed{S = \frac{1}{\sqrt{5}}\left[\frac{1}{4}(4)^2 - 2(4) + 10\right] = \frac{6}{\sqrt{5}} \text{ unit length}}$

Example (8)

A window in the shape of rectangle sermounted by a semi circle. If the perimter of the window is 50 cm , then find the ratio between the two dimensions of the rectangle such that the surface area of the window is to be maximum .

Answer

In this problem, we have to make a relation between surface area and perimeter



Let the two dimensions of the rectangle are x , y cm

and then the diameter of the semi circle will be x too

The perimeter of the window is :

perimeter of the rectangle + perimeter of the semi circle

$$\therefore\ 2y + x + \frac{\pi x}{2} = 50\ (\times 2) \Rightarrow 4y + 2x + \pi x = 100$$

$$\therefore\ 4y = 100 - 2x - \pi x \Rightarrow \text{ by dividing by } 4: \quad \boxed{\therefore\ y = 25 - \frac{1}{2}x - \frac{1}{4}\pi x ---(1)}$$

$\because$ The surface area of the window (S) = area of the rectangle + area of the semi circle

$$\boxed{\therefore\ S = xy + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 ---(2)}\text{ , by substituting } (1) \text{ in } (2):$$

$$\therefore\ S = x\left(25 - \frac{1}{2}x - \frac{1}{4}\pi x\right) + \frac{1}{2}\pi\left(\frac{x}{2}\right)^2 = 25x - \frac{1}{2}x^2 - \frac{1}{4}\pi x^2 + \frac{1}{8}\pi x^2 = 25x - \frac{1}{2}x^2 - \frac{1}{8}\pi x^2$$

$$\therefore\ S' = 25 - x - \frac{1}{4}\pi x \Rightarrow \text{ when } S' = 0 \Rightarrow 25 - x\left(1 + \frac{\pi}{4}\right) = 0 \Rightarrow \therefore x\left(\frac{4}{4} + \frac{\pi}{4}\right) = 25$$

$$\therefore\ x\left(\frac{4+\pi}{4}\right) = 25 \Rightarrow \text{ then } \boxed{x = \frac{100}{4+\pi}}$$

$$\text{And } \because\ S'' = -1 - \frac{1}{4}\pi = -\left(1 + \frac{1}{4}\pi\right) < 0 \Rightarrow \quad \therefore\ S \text{ is maximum at } x = \frac{100}{4+\pi}$$

$$\text{Then by substituting in } (1):\ y = 25 - \frac{1}{2}\left[\frac{100}{4+\pi}\right] - \frac{1}{4}\pi\left[\frac{100}{4+\pi}\right] = 25 - \frac{50}{4+\pi} - \frac{25\pi}{4+\pi}$$

$$\therefore\ y = \frac{25(4+\pi)}{4+\pi} - \frac{50}{4+\pi} - \frac{25\pi}{4+\pi} = \frac{100 + 25\pi - 50 - 25\pi}{4+\pi} = \frac{50}{4+\pi} \Rightarrow \therefore\ \boxed{x : y = 2 : 1}$$

Example (9)

The lengths of three sides of a trapezium are equal and each of length 22 cm . Find the length of the fourth side such that the surface area of the trapezium is to be maximum .

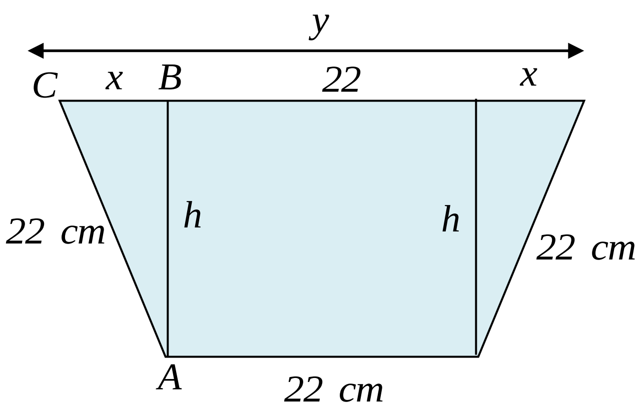
Answer

In this problem, we have to make a relation between Surface area and the length of the sides

let the length of the fourth side be y cm where $y = 22 + 2x - - - (1)$

And $\because$ area of the trapezuim $= \frac{1}{2}(\text{sum of two parallel bases}) \times h$

$$\therefore \quad S = \frac{1}{2}(22 + 22 + 2x)h = \frac{1}{2}(44 + 2x)h - - - (2)$$



In Δ ABC : $\because \; h^2 + x^2 = (22)^2$

$$\therefore \; h = \sqrt{484 - x^2} - - - (3)$$

By substituting (3) in (2)

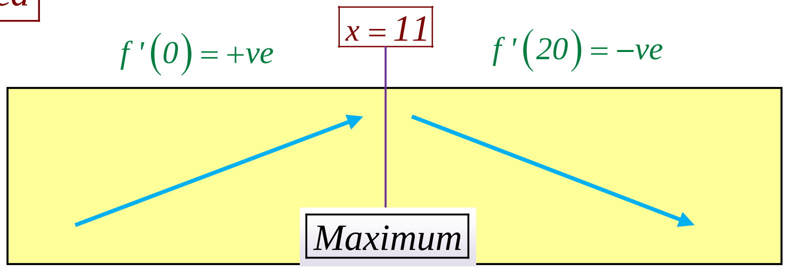
$$\therefore \quad S = \frac{1}{2}(44 + 2x)\sqrt{484 - x^2} = (22 + x)(484 - x^2)^{\frac{1}{2}}$$

$$\therefore \quad S' = \frac{-(22 + x)x}{\sqrt{484 - x^2}} + \sqrt{484 - x^2} = \frac{-22x - x^2 + 484 - x^2}{\sqrt{484 - x^2}} \Rightarrow \therefore \; S' = \frac{484 - 22x - 2x^2}{\sqrt{484 - x^2}}$$

When $S' = 0 \quad \Rightarrow \quad \therefore \; 484 - 22x - 2x^2 = 0 \quad (\div -2)$

$\therefore \quad x^2 + 11x - 242 = 0 \quad \Rightarrow \quad (x - 11)(x + 22) = 0$

$\therefore \quad x = 11$ or $x = -22$ refused



Then the maximum area of the trapezoid occurs when $x = 11$.

By substituting in (1):

The fourth side is $y = 22 + 2(11) = 44$ cm .

Then $h = \sqrt{484 - (11)^2} = 11\sqrt{3} \Rightarrow \therefore \; A_{max} = (22 + 11)(11\sqrt{3}) = 363\sqrt{3}\ cm^2$

Example (10)

Find a point on the curve $y^2 - 2y + 4x - 23 = 0$, *if the distance between it and the point* $(3,1)$ *is to be minimum .*

Answer

In this problem, we have to make a relation between point and distance

Let the required point is $(a\ ,b) \Rightarrow\ \therefore\ b^2 - 2b + 4a - 23 = 0$

$$\therefore\ b^2 - 2b + 1 = 24 - 4a \quad ---(1)$$

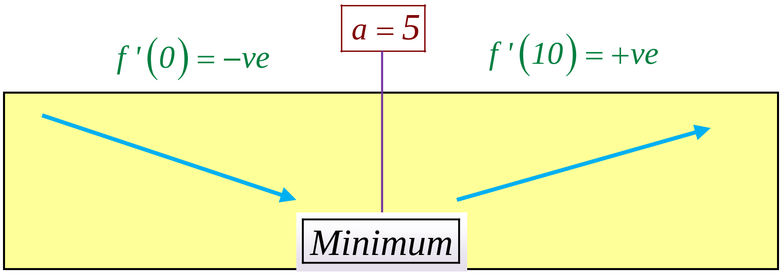
D [*The distance between* $(a\ ,b)$ *and* $(3\ ,1) = \sqrt{(a-3)^2 + (b-1)^2}$]

$\therefore\ D = \sqrt{a^2 - 6a + 9 + b^2 - 2b + 1} \quad ---(2)$ *then substitute* (1) *in* (2)

$$\therefore\ D = \sqrt{a^2 - 6a + 9 + 24 - 4a} = \sqrt{a^2 - 10a + 33} = (a^2 - 10a + 33)^{\frac{1}{2}}$$

$$\therefore\ D' = \frac{1}{2}(a^2 - 10a + 33)^{-\frac{1}{2}}(2a - 10) = \frac{2a - 10}{2\sqrt{a^2 - 10a + 33}} = \frac{\cancel{2}(a-5)}{\cancel{2}\sqrt{a^2 - 10a + 33}} = \frac{(a-5)}{\sqrt{a^2 - 10a + 33}}$$

When $D' = 0 \Rightarrow \boxed{\therefore\ a = 5}$



Then the minimum distance between the two points occurs when $a = 5$.

By substituting in (1):

$\therefore\ b^2 - 2b + 1 = 24 - 4(5)$

$\therefore\ b^2 - 2b - 3 = 0 \Rightarrow (b-3)(b+1) = 0 \Rightarrow \therefore\ \boxed{b = 3}$ *Or* $\boxed{b = -1}$

$\therefore\ (5\ ,3), (5\ ,-1)$ *are the nearest points to the point* $(3\ ,1)$

Example (12)

$ABCD$ is a rectangle in which $AB = 8$ cm, and $BC = 12$ cm. If $H \in \overline{AB}$ and $O \in \overline{BC}$ such that $BO = 4AH = x$. Then prove that surface area of $\Delta\ DHO$ is not less than $43.5\ cm^2$.

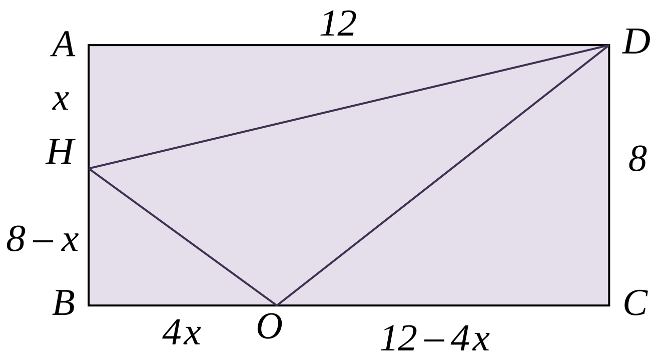
Answer

In this problem, we have to make a relation between Area of triangle and the length of rectangle sides $\Rightarrow$ so let $AH = x$ and $BO = 4x$

$\because$ Area $\Delta\ DHO$ = Area $\square\ ABCD$ – Area $\Delta\ AHD$ – Area $\Delta\ BHO$ – Area $\Delta\ OCD$

$$\text{Area } \Delta\ DHO = 12\times 8 - \frac{1}{2}\times 12x - \frac{1}{2}\times 4x(8-x) - \frac{1}{2}\times 8(12-4x) = 96 - 6x - \cancel{16x} + 2x^2 - 48 + \cancel{16x}$$

$\therefore$ Area $\Delta\ DHO = 2x^2 - 6x + 48 ---(1)$



$\therefore\ A' = 4x - 6$, when $A' = 0 \Rightarrow 4x - 6 = 0$

$\therefore\ x = \frac{3}{2} ---(2)$ and $\because\ A'' = 4 > 0$

$\therefore\ \Delta\ DHO$ is minimum when $x = \frac{3}{2}$

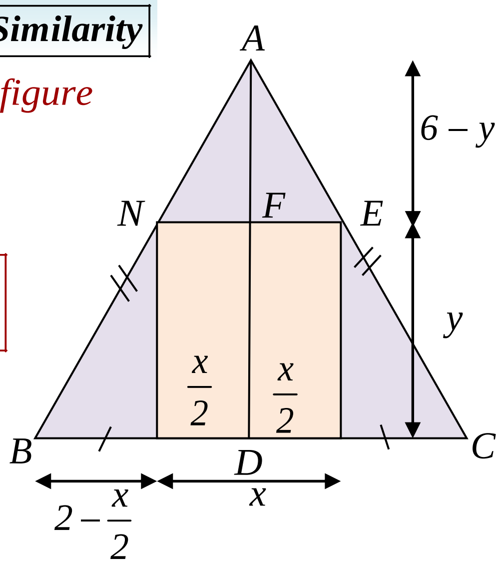
$\therefore$ By substituting (2) in (1), the minimum surface area of $\Delta\ DHO = 2\left(\frac{3}{2}\right)^2 - 6\left(\frac{3}{2}\right) + 48 = 43.5$

Example (13)

An isosceles triangle, Its base is of length 4 cm and its height is of length 6 cm, find the maximum surface area of the rectangle which can be drawn inside the triangle.

Answer

In any triangle, if a line is parallel to its base, think of using **Similarity**



Let the two dimensions of the rectangle are x, y as in the figure

$\because\ \overline{NF} \parallel \overline{BD} \Rightarrow \therefore\ \Delta AFN \sim \Delta ADB \Rightarrow \therefore\ \frac{AF}{AD} = \frac{NF}{DB}$

$\therefore\ \frac{6-y}{6} = \frac{\frac{x}{2}}{2} \Rightarrow \therefore\ 12 - 2y = 3x \Rightarrow$ $y = 6 - \frac{3}{2}x ---(1)$

$\because\ S = xy = x\left(6 - \frac{3}{2}x\right) = 6x - \frac{3}{2}x^2$

$\therefore\ \frac{dS}{dx} = 6 - 3x \Rightarrow$ when $\frac{ds}{dx} = 0 \Rightarrow$ $\therefore\ x = 2$

And $\because\ \frac{d^2s}{dx^2} = -3 < 0 \Rightarrow \therefore$ the surface area is maximum when $x = 2$

$\therefore$ The maximum surface area $= 6\times 2 - \frac{3}{2}\times 4 = 12 - 6 = 6\ cm^2$.