Example (2)

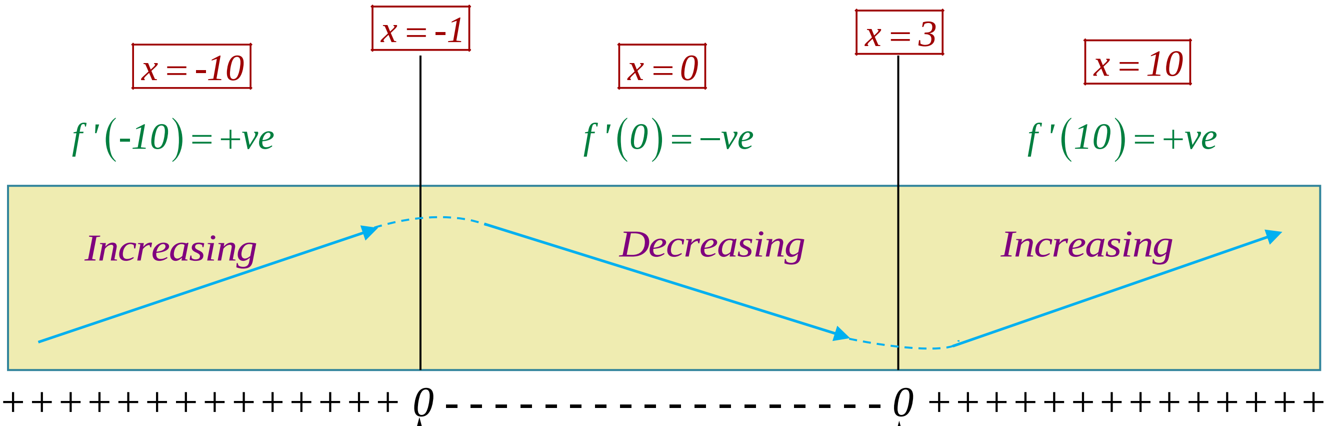
Determine the increasing and decreasing intervals of the function : $f(x) = x^3 - 3x^2 - 9x + 1$

Answer

$\because$ $f(x)$ is differentiable and continous on R

(1) $f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1) \Rightarrow$ (2) for $f'(x) = 0$

$\therefore$ $\boxed{x = 3}$ and $\boxed{x = -1}$ are the critical points



$f(x)$ is increasing on $]-\infty, -1[$

$f(x)$ is decreasing over $]-1, 3[$

$f(x)$ is increasing over $]3, \infty[$

$f(x)$ has a local maximum at $x = -1$

$f(x)$ has a local minimum at $x = 3$

Example (3)

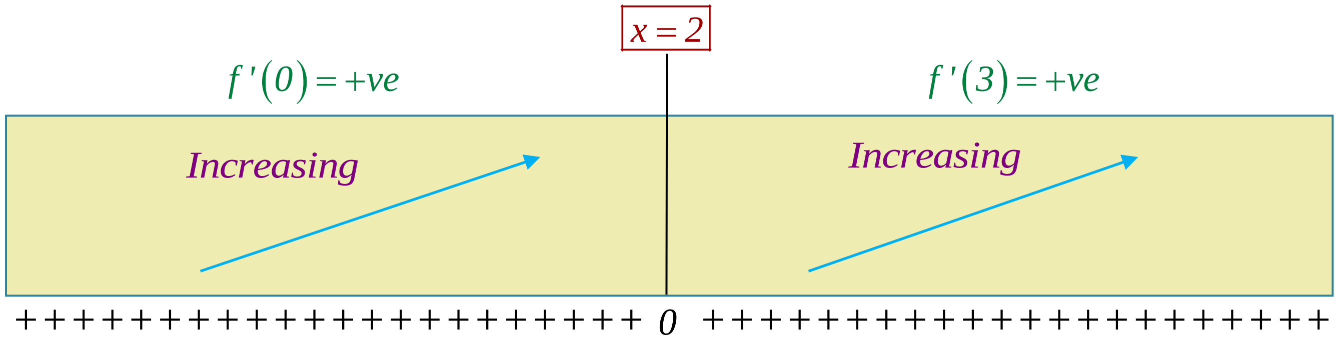
Discuss the monotony of the function : $f(x) = x^3 - 6x^2 + 12x - 5$

Answer

$\because$ $f(x)$ is differentiable and continous on R

(1) $f'(x) = 3x^2 - 12x + 12 = 3(x^2 - 4x + 4) = 3(x-2)^2$

(2) $f'(x) = 0 \Rightarrow 3(x-2)^2 = 0 \Rightarrow$ $\boxed{x = 2}$ is the only critical point



$f(x)$ is increasing over on $]-\infty, 2[$

$f(x)$ is increasing over on $]2, \infty[$

Neither local maximum
Nor local minimum

Then $x = 2$ is not a critical point as the sign of curvature didn't change

Example (4)

Determine the intervals of increasing and decreasing of : $f(x) = x - e^x$

Answer

$\because f(x)$ is continous and differentiable

(1) $f'(x) = 1 - e^x$ (2) $f'(x) = 0 \Rightarrow 1 - e^x = 0 \Rightarrow e^x = 1 \Rightarrow \boxed{\therefore x = 0}$

$\boxed{x = 0}$

$f'(-1) = +ve$ $f'(2) = -ve$

Increasing Decreasing

+ 0 -

$f(x)$ is increasing over on $]-\infty, 0[$ $f(x)$ is decreasing over on $]0, \infty[$

local maximum at $x = 0$

Example (5)

Discuss the monotony of the function : $f(x) = x^3(x-1)^2$

Answer

....

....

....

....

....

....

$x =$ $x = —$ $x =$

$f'(\quad) =$ $f'(\quad) =$ $f'(\quad) =$ $f'(\quad) =$

....

Then the critical points are ....

While $x =$ is not a critical point ....

Example (6)

Determine the increasing and decreasing intervals of the function : $f(x)=|2x-3|-x+2$

Answer

$$f(x)=\begin{cases}2x-3-x+2 & , x\ge\frac{3}{2}\\ 3-2x-x+2 & , x<\frac{3}{2}\end{cases} \Rightarrow f(x)=\begin{cases}x-1 & , x\ge\frac{3}{2}\\ 5-3x & , x<\frac{3}{2}\end{cases}$$

$\because f(x)$ is continous at $x=\frac{3}{2}$ as $f(3)=f(3^-)=f(3^+)=\frac{1}{2}$

And $\because f(x)$ is a double function, we have to get the critical points from:

(i) Check if : $f'\left(\frac{3}{2}^+\right)\ne f'\left(\frac{3}{2}^-\right)$ by using defferentiability $\qquad (ii)$ $f'(x)=0$

$$(i)\text{For } f'\left(\frac{3}{2}^+\right): \lim_{h\to0^+}\frac{f\left(\frac{3}{2}+h\right)-f\left(\frac{3}{2}\right)}{h} \Rightarrow \lim_{h\to0^+}\frac{\left(\frac{3}{2}+h\right)-1-\left[\frac{3}{2}-1\right]}{h}$$

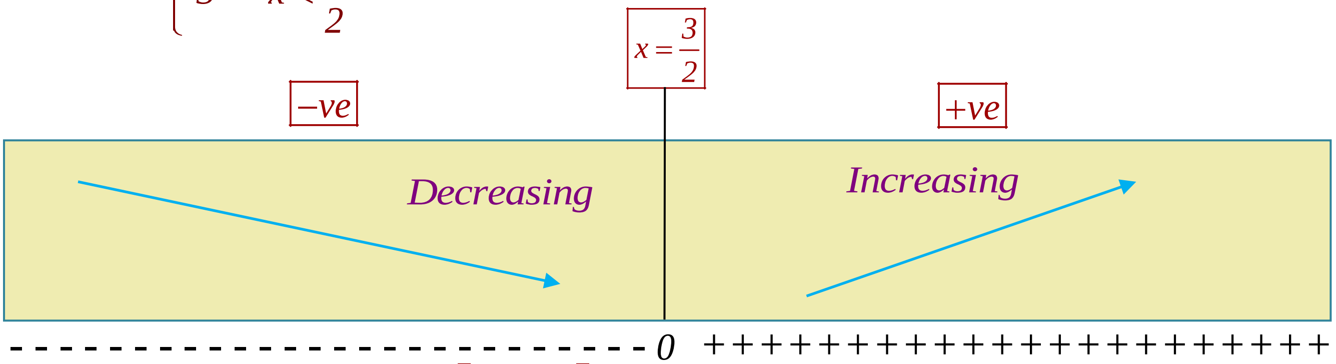
$$\therefore \lim_{h\to0^+}\frac{\frac{3}{2}+h-1-\frac{3}{2}+1}{h} \Rightarrow \lim_{h\to0^+}\frac{h}{h} \Rightarrow \lim_{h\to0^+} 1=1$$

$$\text{For } f'\left(\frac{3}{2}^-\right): \lim_{h\to0^+}\frac{f\left(\frac{3}{2}+h\right)-f\left(\frac{3}{2}\right)}{h} \Rightarrow \lim_{h\to0^+}\frac{\left(5-3\left(\frac{3}{2}+h\right)\right)-\left[5-\frac{3(3)}{2}\right]}{h}$$

$$\therefore \lim_{h\to0^-}\frac{5-\frac{9}{2}-3h-5+\frac{9}{2}}{h} \Rightarrow \lim_{h\to0^-}\frac{-3h}{h} \Rightarrow \lim_{h\to0^-} -3=-3$$

So $\because f'\left(\frac{3}{2}^+\right)\ne f'\left(\frac{3}{2}^-\right)$ doesn't exist $\Rightarrow$ $\boxed{x=\frac{3}{2}}$ is a critical point

(ii) $\because f'(x)=\begin{cases}1 & x\ge\frac{3}{2}\\ -3 & x<\frac{3}{2}\end{cases}$, so we can't find $f'(x)=0 \Rightarrow \boxed{x=\frac{3}{2}}$ is the only critical point



$f(x)$ is decreasing over on $\left]-\infty,\frac{3}{2}\right[$ $\qquad$ $f(x)$ is increasing over on $\left]\frac{3}{2},\infty\right[$

Local minimum

Example (7)

Determine the increasing and decreasing intervals of the function : $f(x)=x|x-2|$

Answer

$$f(x)=\begin{cases} x(x-2) & x\geq 2 \\ -x(x-2) & x<2 \end{cases} \Rightarrow f(x)=\begin{cases} x^2-2x & , x\geq 2 \\ 2x-x^2 & , x<2 \end{cases}$$

$\because f(x)$ is continous at $x=2$ as $f(2)=f(2^-)=f(2^+)=0$

$\because f(x)$ is a double function, we have to get the critical points from:

(i) Check if : $f'(2^+)\neq f'(2^-)$ by using defferentiability $\qquad (ii)\ f'(x)=0$

$$(i)\text{For } f'(2^+):\ \lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} \Rightarrow \lim_{h\to 0^+}\frac{2(2+h)-2-[2(2)-2]}{h}$$

$$\therefore \lim_{h\to 0^+}\frac{4+2h-2-4+2}{h} \Rightarrow \lim_{h\to 0^+}\frac{2h}{h} \Rightarrow \lim_{h\to 0^+} 2=2$$

$$\text{For } f'(2^-):\ \lim_{h\to 0^+}\frac{f(2+h)-f(2)}{h} \Rightarrow \lim_{h\to 0^+}\frac{2-2(x+2)-[2-2(2)]}{h}$$

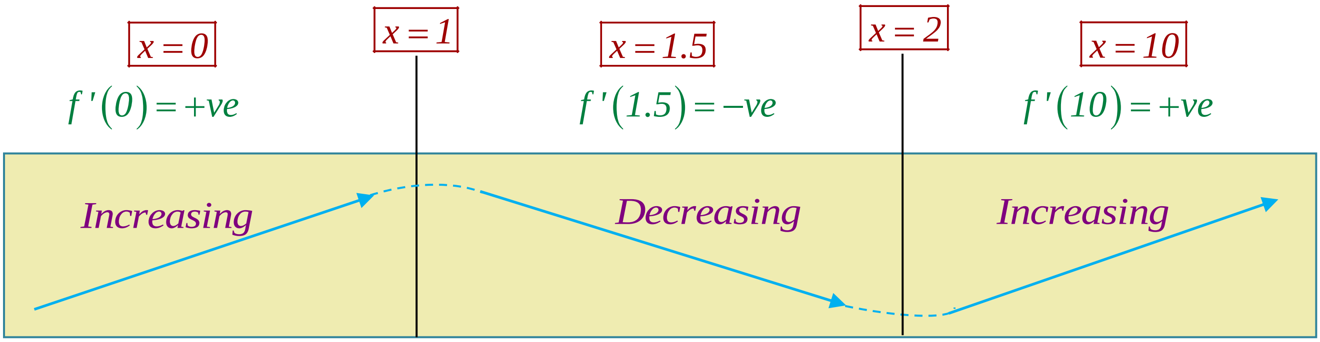
$$\therefore \lim_{h\to 0^-}\frac{2-4-2h-2+4}{h} \Rightarrow \lim_{h\to 0^-}\frac{-2h}{h} \Rightarrow \lim_{h\to 0^-} -2=-2$$

So $\because f'(2^+)\neq f'(2^-)$ doesn't exist $\Rightarrow$ $\boxed{x=2}$ is the a critical point

$$(ii)\ \because f'(x)=\begin{cases} 2x-2 & x\geq 2 \\ 2-2x & x<2 \end{cases}$$

For $x\geq 2$: $2x-2=0 \Rightarrow \boxed{x=1}$ is a critical point but $x\geq 2$, $\boxed{\text{so } x=1 \text{ is refused}}$

For $x<2$: $2-2x=0 \Rightarrow \boxed{x=1}$ is a critical point



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$f(x)$ is increasing on $]-\infty,1[$

$f(x)$ is decreasing over $]-1,2[$

$f(x)$ is increasing over $]2,\infty[$

$f(x)$ has a local maximum at $x=1$

$f(x)$ has a local minimum at $x=2$

Example (8)

Find the intervals where the following function is increasing or decreasing where :

$$f(x)=Sin\,x(1+Cos\,x)\ \ in\ \left[0,\frac{\pi}{2}\right]$$

Answer

$\because\ f(x)$ is continous on $\left[0,\frac{\pi}{2}\right]$

$f'(x)=Sin\,x(-Sin\,x)+(1+Cos\,x)Cos\,x \Rightarrow f'(x)=-Sin^2x+Cos\,x+Cos^2x$

Remember that : $Sin^2x=1-Cos^2x \Rightarrow f'(x)=-(1-Cos^2x)+Cos\,x+Cos^2x$

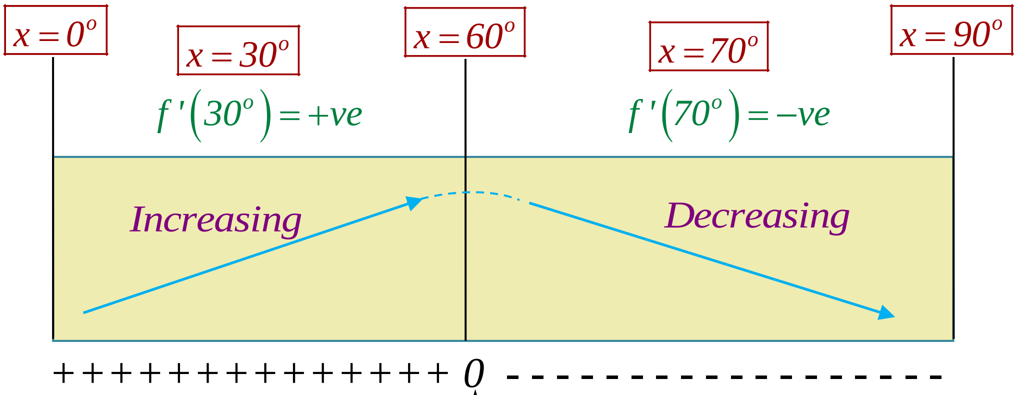
$\therefore\ f'(x)=2Cos^2x+Cos\,x-1 \Rightarrow$ for $f'(x)=0 \Rightarrow (2Cos\,x-1)(Cos\,x+1)=0$

$\therefore\ Cos\,x=\frac{1}{2}$ and $Cos\,x=-1$

Then $\boxed{x=Cos^{-1}\frac{1}{2}=60^o}$ | $\boxed{x=180^o\ \text{"refused"}}$

Or $\boxed{x=360^o-60^o=300^o\ \text{"refused"}}$

$\therefore$ The only critical point is $\boxed{x=60^o}$



$f(x)$ is increasing on $]0, 60^o[$

$f(x)$ is decreasing over $]60^o, 90^o[$

$f(x)$ has a local maximum at $x=60^o$

Then $x=60^o$ is a critical point as the sign of curvature changed

Example (9)

Find the intervals where the following function is increasing or decreasing where :

$$f(x) = 2Sin\,x\,Cos\,x \quad in \quad [0, \pi]$$

Answer

....

....

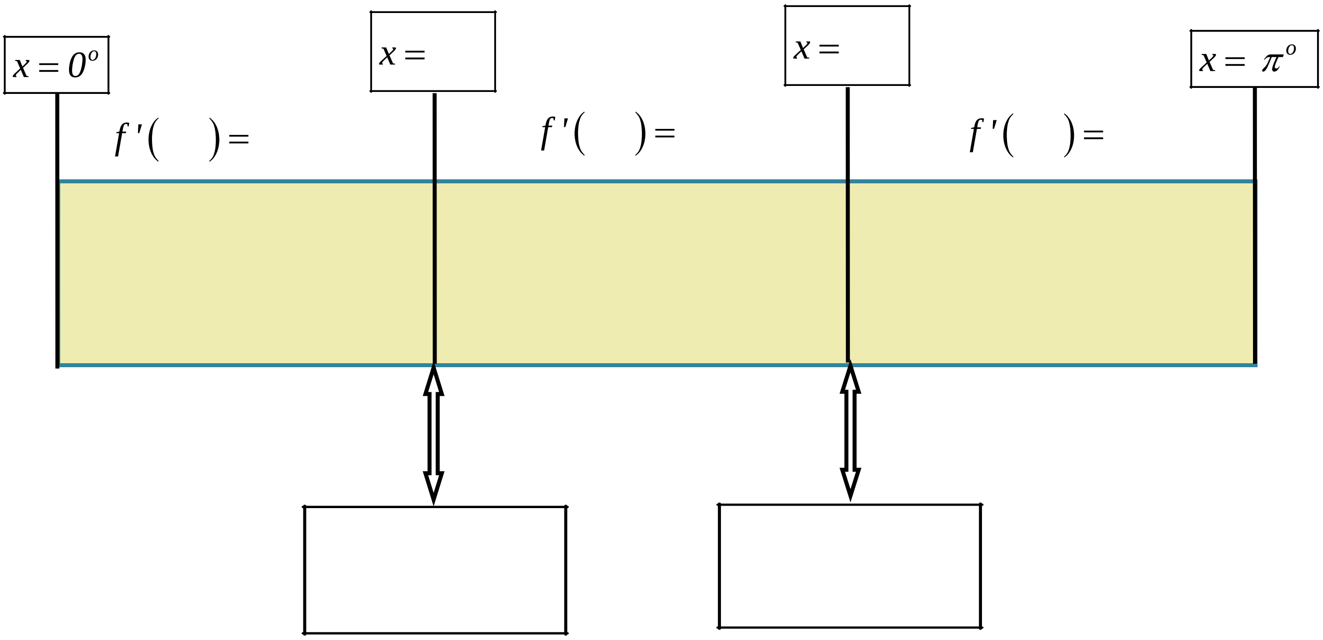
....

....

....

....

....



....

Then the critical points are ....

Special cases of increasing and decreasing functions problems
"Fractional functions"

Remember that

In any fractional function, critical points occurs if:

$(1)\ f'(x)$ *exists and equals to zero ." Numerator = 0"*

$(2)\ f'(x)$ *doesn't exist " undefined when denomenator = 0"*

* *So, the first step to solve the fractional problems is to* ***find the domain***

* *Then determine the increase and decrease intervals over the domain*

Example (10)

Investigate the monotony of the function : $f(x) = \dfrac{x}{x-2} \quad , \quad x \neq 2$

Answer

$\because\ f(x)$ *is a fraction function* $\Rightarrow \therefore$ *the domain is* $R - \{2\}$

$\therefore\ f'(x) = \dfrac{x-2-x}{(x-2)^2} = \dfrac{-2}{(x-2)^2}$ *(negative value)*

(1) *For* $f'(x) = 0 \Rightarrow \therefore\ \dfrac{-2}{(x-2)^2} = 0$

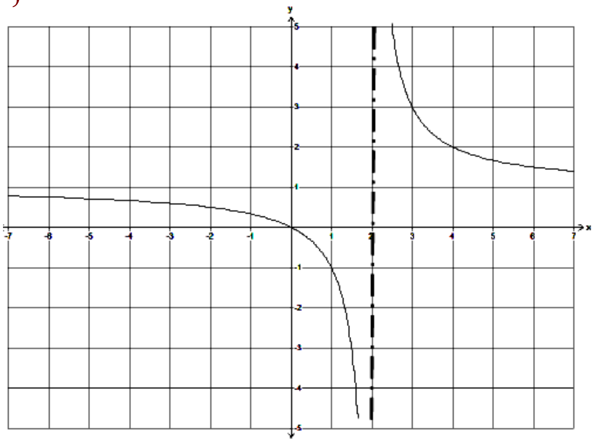
Numerator = 0 $\Rightarrow\ -2 = 0$ *(can't be)*

$(2)\ f'(x)$ *is undefined when denominator = 0*

$\therefore\ (x-2)^2 = 0 \Rightarrow$ *then* $x = 2$ *which is refused in the domain*

So there is *no* *critical point in this function .*

Then the $f(x)$ *is always decreasing on* $R - \{2\}$



$x = 2$

$f'(0) = -ve$ | $f'(3) = -ve$

Decreasing | Decreasing

decreasing on $]-\infty, 2[$ | decreasing on $]2, \infty[$

Example (11)

Determine the increasing and decreasing intervals of the function : $f(x)=\frac{1}{3}x-\sqrt[3]{x}$

Answer

First The domain of $f(x)$ is R

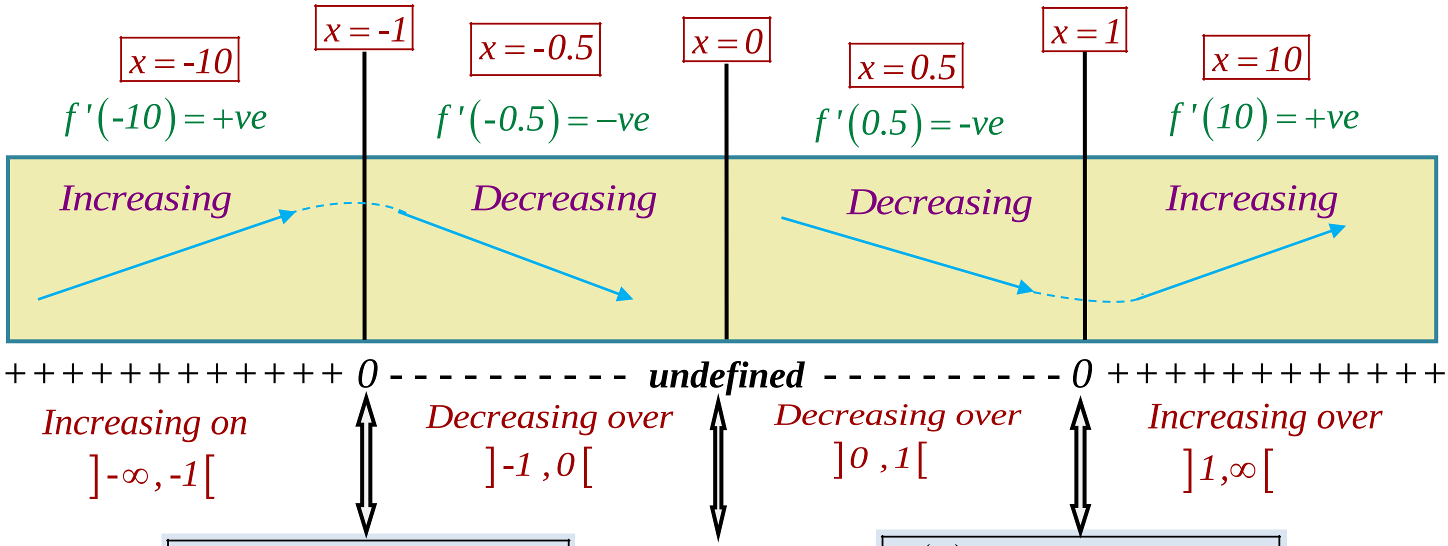
$$f(x)=\frac{1}{3}x-x^{\frac{1}{3}} \Rightarrow \therefore f'(x)=\frac{1}{3}-\frac{1}{3}x^{\frac{-2}{3}}=\frac{1}{3}\left[1-\frac{1}{x^{\frac{2}{3}}}\right]=\frac{1}{3}\left[\frac{x^{\frac{2}{3}}-1}{x^{\frac{2}{3}}}\right] \quad \text{where } x\neq 0$$

(1) For $f'(x)=0 \Rightarrow \therefore \frac{1}{3}\left[\frac{x^{\frac{2}{3}}-1}{x^{\frac{2}{3}}}\right]=0 \Rightarrow \therefore \frac{x^{\frac{2}{3}}-1}{x^{\frac{2}{3}}}=0$

Numerator $=0 \Rightarrow \therefore x^{\frac{2}{3}}=1 \Rightarrow$ $\boxed{\therefore x=\pm 1 \text{ (agreed)}}$

(2) $f'(x)$ is undefined when denominator $=0 \Rightarrow \therefore x^{\frac{2}{3}}=0 \Rightarrow$ $\boxed{\therefore x=0 \in \text{domain (agreed)}}$

So we can say that the critical points are : $\boxed{x=1}$ And $\boxed{x=-1}$ And $\boxed{x=0}$



$f(x)$ has local maximum at $x=-1$

$f(x)$ has local minimum at $x=1$

Then $x=1$ and $x=-1$ are critical points , while $x=0$ is not a critical point

Example (12)

Dicuss the monotony of the function : $f(x)=3-\ln x^2$

Answer

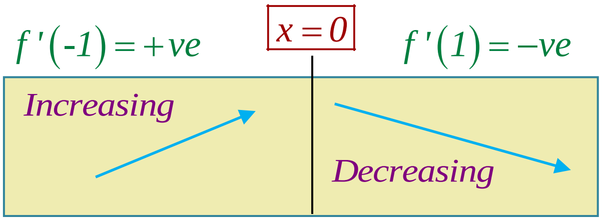
The domain is $R-\{0\}$ $\Rightarrow$ $f(x)$ is continous on $R-\{0\}$ and $\because\ f'(x)=\frac{-2x}{x^2}=\frac{-2}{x}$

(1) $f'(x)=0 \Rightarrow \therefore -2=0 \Rightarrow$ can't be

(2) $f'(x)$ is undefined at $x=0$ (refused)

$\therefore$ From (1) and (2):

There is no critical points of f



Increasing on $]-\infty,0[$ — decreasing on $]0,\infty[$

Example (13)

Dicuss the monotony of the function : $f(x)=2\ln x-x^2$

Answer

The domain is $]0,\infty[$ $\Rightarrow$ $f(x)$ is continous on $]0,\infty[$

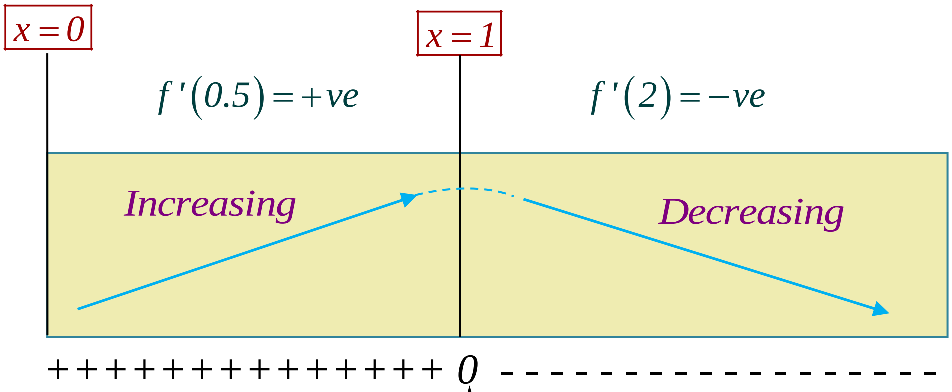
$\because\ f'(x)=\frac{2}{x}-2x \Rightarrow \therefore\ f'(x)=\frac{2-2x^2}{x}$

(1) $f'(x)=0 \Rightarrow \therefore 2-2x^2=0 \Rightarrow \therefore 2x^2=2 \Rightarrow \therefore x^2=1$

$\therefore x=1$ (agreed) or $x=-1$ (refused)

(2) $f'(x)$ is undefined at $x=0$ (refused)

$\therefore$ From (1) and (2): there is one critical point $x=1$



$f(x)$ is increasing on $]0,1[$

$f(x)$ is decreasing over $]1,\infty[$

$f(x)$ has a local maximum at $x=1$

Then $x=1$ is a critical point as the sign of curvature changed

Case (2) : Local maximum and local minimum "Classification"

We can get local maximum or local minimum by two ways , either :

(1) By using 1^{st} derivative test "as we have done before in the previous case"

Note : we use this method if the function was linear " its power is one "

(2) By using 2^{nd} derivative test

Note : we use this method if he asked me to find local maximum or minimum directly.

Steps

(1) Find by differentiation the first derivative of the given function $f'(x)$.

(2) Find the critical points using $f'(x) = 0$, you will get the value of x for this critical point.

(3) Get the second derivative .

(4) Substitute the critical points if exists in the second derivatives .

(5) (i) If the result is + ve …… Then it is local minimum.

(ii) If the result is - ve …… Then it is local maximum.

(iii) If the result is zero …… Then it is Neither local minimum Nor local maximum

(6) Substitute the critical point is the original function to get the point

---

Example (1)

Find the local maximum and the local minimum of $f(x) = x^4 - 18x^2$

Answer

$f(x)$ is continous on R

$f'(x) = 4x^3 - 36x = 4x(x^2 - 9) = 4x(x-3)(x+3)$

For $f'(x) = 0 \Rightarrow \boxed{x = 0}$ and $\boxed{x = 3}$ and $\boxed{x = -3}$

$\therefore \quad f''(x) = 12x^2 - 36$

$\therefore \quad f''(0) = -36 < 0 \Rightarrow \boxed{\therefore \text{ at } x = 0} \rightarrow$ the function has a local maximum

* After substituting in the original function $\Rightarrow$ the local maximum point is $\boxed{(0, 0)}$

$f''(3) = 72 > 0 \Rightarrow \boxed{\therefore \text{ at } x = 3} \rightarrow$ the function has a local minimum

* After substituting in the original function $\Rightarrow$ the local maximum point is $\boxed{(3, -81)}$

$f''(-3) = 72 > 0 \Rightarrow \boxed{\therefore \text{ at } x = -3} \rightarrow$ the function has a local minimum

* After substituting in the original function $\Rightarrow$ the local maximum point is $\boxed{(-3, -81)}$

Example (2)

Classify the function f where $f(x)=3x^5+2x^3-1$

Answer

$f(x)$ is continous on R

$\therefore\ f'(x)=15x^4+6x^2=3x^2(5x^2+2)$, for $f'(x)=0$

$\therefore\ 3x^2(5x^2+2)=0 \Rightarrow \boxed{x=0}$ or $x^2=\frac{-2}{5}$ "refused"

$f''(x)=60x^3+12x$

$f''(0)=0 \quad \Rightarrow$ Neither local miaximum Nor minimum

---

Example (3)

Investigate the local maximum and the local minimum of $f(x)=\frac{1-x^2}{1+x^2}$

Answer

$\because$ The problem is fraction $\Rightarrow \therefore$ the domain of $f(x)$ is R and $f(x)$ is continous on R

To get the critical points: $y'=\frac{-2x(1+x^2)-2x(1-x^2)}{(1+x^2)^2}=\frac{-4x}{(1+x^2)^2}$

(1) For $f'(x)=0 \Rightarrow \therefore \frac{-4x}{(1+x^2)^2}=0 \Rightarrow \therefore -4x=0 \Rightarrow \boxed{\therefore\ x=0}$

(2) $f'(x)$ is undefined when $(1+x^2)^2=0 \Rightarrow$ here x has no real solution

Then the only critical point is at $x=0$

$$\therefore\ y''=\frac{-4(1+x^2)^2+4x(1+x^2)4x}{(1+x^2)^4}$$

At $\boxed{x=0} \Rightarrow y''=\frac{-4(1)^2}{(1)^4}=-4<0 \Rightarrow$ the function has a local maximum at $x=0$

Then the local maximum point is $(0,1)$

Example (4)

Classify: $y = x + \dfrac{4}{x-1} \quad , \; x \neq 1$

Answer

The domain of $f(x)$ is $R-\{1\} \Rightarrow f(x)$ is continous on $R-\{1\}$

$$\because f'(x) = 1 - \frac{4}{(x-1)^2} = \frac{(x-1)^2 - 4}{(x-1)^2}$$

(1) For $f'(x) = 0 \Rightarrow \dfrac{(x-1)^2-4}{(x-1)^2} = 0 \Rightarrow \therefore (x-1)^2 - 4 = 0 \Rightarrow (x-1)^2 = 4$

$\therefore x - 1 = \pm 2 \Rightarrow \therefore \boxed{x=3}$ or $\boxed{x=-1}$ are critical points

(2) Note : $\boxed{x=1}$ is not a critical point as it is not defined in the domain of $f(x)$

And $\because f''(x) = \dfrac{8}{(x-1)^3} \Rightarrow \therefore f''(3) = \dfrac{8}{8} = 1 > 0 \Rightarrow \therefore f(x)$ is a local minimum point at $x = 3$

And $f''(-1) = \dfrac{8}{-8} = -1 < 0 \Rightarrow \therefore f(x)$ is a local maximum point at $x = -1$

Example (5)

Find the local maximum and the local minimum points of $f(x) = \begin{cases} 2x + x^2 & x \leq 0 \\ 2x - x^2 & x > 0 \end{cases}$

Answer

$\because f(x)$ is continous at $x = 0$ as $f(0) = f(0^-) = f(0^+) = 0$

$\because f(x)$ is a double function, we have to get the critical points from:

(i) Check if : $f'(0^+) \neq f'(0^-)$ by using defferentiability $\qquad (ii)$ $f'(x) = 0$

(i) For $f'(0^+)$: $\lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} \Rightarrow \lim\limits_{h \to 0^+} \dfrac{2h + h^2 - 0}{h} \Rightarrow \lim\limits_{h \to 0^+} \dfrac{h(2+h)}{h} = 2$

For $f'(0^-)$: $\lim\limits_{h \to 0^+} \dfrac{f(0+h) - f(0)}{h} \Rightarrow \lim\limits_{h \to 0^+} \dfrac{2h - h^2 - 0}{h} \Rightarrow \lim\limits_{h \to 0^+} \dfrac{h(2-h)}{h} = 2$

So $\because f'(0^+) = f'(0^-) \Rightarrow \boxed{\therefore x = 0}$ is not a critical point

$(ii) \because f'(x) = \begin{cases} 2 + 2x & , \; x \leq 0 \\ 2 - 2x & , \; x > 0 \end{cases}$

For $x \leq 0 \Rightarrow \therefore 2 + 2x = 0 \Rightarrow \boxed{x = -1 \,(agreed)}$

For $x > 0 \Rightarrow 2 - 2x = 0 \Rightarrow \boxed{x = 1 \,(agreed)} \Rightarrow \boxed{\therefore x = \pm 1 \text{ are critical points}}$

$\therefore$ For $x \leq 0$: $\because f''(x) = 2 \Rightarrow \therefore f''(-1) = 2 > 0$

$\therefore$ **$x = -1$ is a local minimum of the function** $\Rightarrow$ its point is $(-1,-1)$

And for $x > 0$: $f''(x) = -2 \Rightarrow \therefore f''(1) = -2 < 0$

$\therefore$ **$x = 1$ is a local maximum of the function** $\Rightarrow$ its point is $(1,1)$

Example (6)

Prove that the function $f(x)=|x-1|$ has a critical point at $x=1$, then show that it is a local minimum of the function

Answer

$$\because \ f(x)=|x-1|=\begin{cases} \text{..........} & , \ x \geq \text{.....} \\ \text{..........} & , \ x < \text{.....} \end{cases}$$

$\because \ f(x)$ is a double function, we have to get the critical points from:

(i) Check if : .... (i) ....

(i) For $f'(1^+)$: $\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow \lim_{h\to 0^+} \frac{\text{..........................}}{h} = \text{.......}$

For $f'(1^-)$: $\lim_{h\to 0^+} \frac{f(1+h)-f(1)}{h} \Rightarrow \lim_{h\to 0^+} \frac{\text{..........................}}{h} = \text{.......}$

So $\because \ f'(1^+) \neq f'(1^-) \Rightarrow$ $\therefore \ x=$ .... is a critical point

$$(ii) \ \because \ f'(x)=\begin{cases} \text{..........} & , \ x \geq \text{.....} \\ \text{..........} & , \ x < \text{.....} \end{cases}$$

For $x \geq$ ..... $\Rightarrow \therefore$ .... (which has no solution)

For $x<1$ $\Rightarrow \therefore$ .... (which has no solution)

for $x \geq 1$ $\Rightarrow f'(1^+)=1>0 \Rightarrow f(x)$ is increasing over $]1,\infty[$

for $x<1$ $\Rightarrow f'(1^-)=-1<0 \Rightarrow f(x)$ is decreasing over $]\infty,1[$

Thus $x=1$ is a local minimum of the function

Note : here we can't use second derivative because $f(x)$ is linear .

Example (7)

Find the values of a and b given that the function : $f(x)=x^2+ax+b$ has a critical point at $x=2$ and $f(2)=1$, hence determine the type of the point $(2,1)$ and whether it is local maximum or local minimum

Answer

$\because \ f(2)=1 \Rightarrow f(2)=(2)^2+2a+b=1 \Rightarrow$ $\therefore \ 2a+b=-3 --- (1)$

The function has a critical point at $x=2$ means that we must find $f'(2)=0$

$f'(x)=2x+a \Rightarrow \therefore \ f'(2)=0 \Rightarrow 4+a=0 \Rightarrow$ $a=-4$

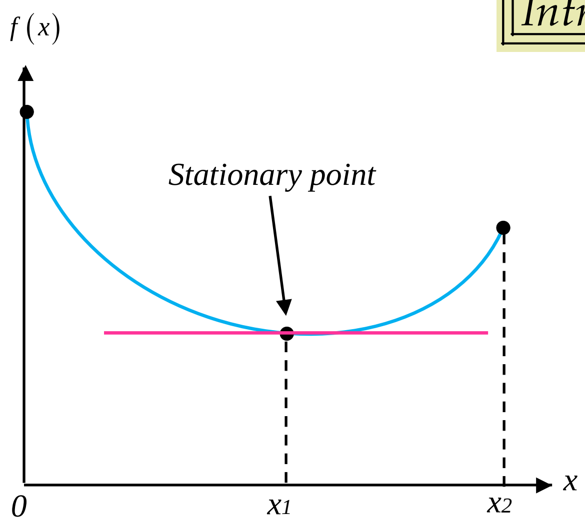
Substitute in (1) : $2(-4)+b=-3 \Rightarrow$ $\therefore \ b=-3+8=5$

Then the function is : $f(x)=x^2-4x+5$

$\therefore \ f''(x)=2 \Rightarrow$ then at $x=2$: $f''(2)=2>0 \Rightarrow$ then $f(x)$ has a local minimum at $x=2$

Case (3): Absolute maximum & minimum

Introduction



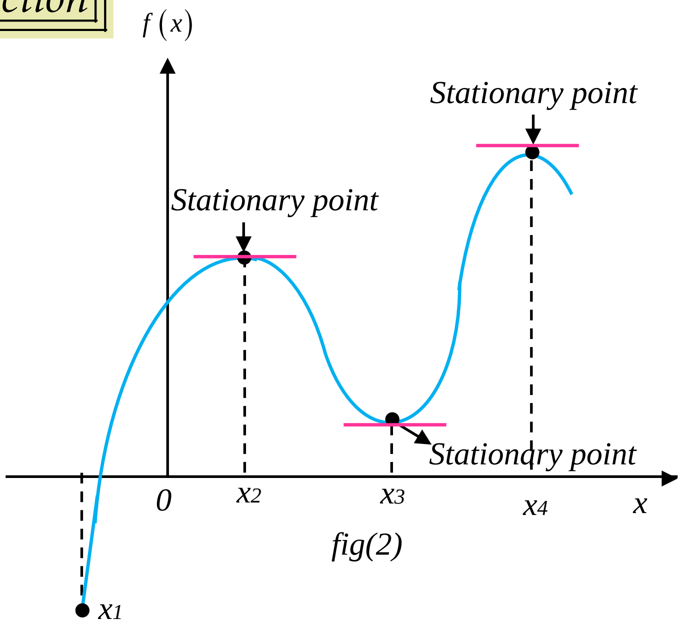
fig(1)

The highest (overall maximum)

$x = 0$

The lowest (overall minimum)

$x = x_1$



fig(2)

(maximum) point occurs at :

$x = x_4$

(minimum) point accures at :

$x = x_1$

Steps

(1) *Differentiate the function* $f(x)$ *and get* $f'(x)$.

(2) *Solve the equation* $f'(x) = 0$ *to find the values of the critical points.*

(3) *Take the critical points which are inside the given interval.*

(4) *Substitute in the original function by using the critical points.*

(5) *Choose the highest value (maximum) and choose the lowest value (minimum).*

Example (1)

Find the absolute maximum and the absolute minimum of : $f(x) = 1 + 12x - x^3$ *where* $x \in [1,3]$

Answer

$\because f(x)$ *is continous on* $[1,3] \Rightarrow f'(x) = 12 - 3x^2 = 3(2-x)(2+x)$

for $f'(x) = 0 \Rightarrow 3(2-x)(2+x) = 0 \Rightarrow \boxed{x = 2}$ *or* $\boxed{x = -2}$

But $-2 \notin [1,3] \Rightarrow \therefore$ *the only critical point at* $\boxed{x = 2}$

$\therefore f(2) = 1 + 24 - 8 = 17$ *and* $f(1) = 1 + 12 - 1 = 12$ *and* $f(3) = 1 + 36 - 27 = 10$

Thus the absolute maximum is 17 *at* $x = 2$

And the absolute minimum is 10 *at* $x = 3$

Example (2)

Determine the absolute maximum and minimum of the function : $f(x) = x^3 - 3x^2 - 9x + 1$ on $[-2,4]$

Answer

$\because f(x)$ is continous on R

$f'(x) = 3x^2 - 6x - 9 = 3(x-3)(x+1) \Rightarrow$ for $f'(x) = 0$

$\therefore$ $\boxed{x=3}$ or $\boxed{x=-1}$ are the Critical points and $\in [-2,4]$

$f(-2) = (-2)^3 - 3(-2)^2 - 9(-2) + 1 = -1$ $f(-1) = (-1)^3 - 3(-1)^2 - 9(-1) + 1 = 6$

$f(3) = (3)^3 - 3(3)^2 - 9(3) + 1 = -26$ $f(4) = (4)^3 - 3(4)^2 - 9(4) + 1 = -19$

Then the absolute maximum point is $(-1,6)$ And the absolute minimum point $(3,-26)$

Example (3)

Find the absolute maximum and absolute minimum of $f(x) = \dfrac{x}{x-1}$ in $[2,4]$

Answer

$\because f(x)$ is continous on $[2,4]$

(1) $\therefore f'(x) = \dfrac{-1}{(x-1)^2} \Rightarrow \therefore$ when $f'(x) = 0 \Rightarrow -1 = 0$ (no real solution)

(2) $f'(x)$ is undefined when denominator $= 0 \Rightarrow (x-1)^2 = 0 \Rightarrow \boxed{\therefore x = 1 \text{ (refused)}}$

$\therefore$ There is no critical points in this function

$\therefore f(2) = \dfrac{2}{2-1} = 2$ and $f(4) = \dfrac{4}{4-1} = \dfrac{4}{3}$

$\therefore$ The absolute maximum is 2 at $x = 2$ and the absolute minimum is $\dfrac{4}{3}$ at $x = 4$

Example (4)

Determine the absolute maximum and minimum of the function : $f(x) = x\,e^{-x}$ over $x \in [0,2]$

Answer

$\because f(x)$ is continous on $[0,2]$

$f'(x) = -x\,e^{-x} + e^{-x} \Rightarrow$ for $f'(x) = 0 \Rightarrow -x\,e^{-x} + e^{-x} = 0$ (divide by e^{-x})

$\therefore -x + 1 = 0 \Rightarrow \therefore$ $\boxed{x=1}$ is the Critical point and $\in [0,2]$

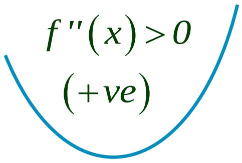
$f(0) = 0$ $f(1) = e^{-1} \simeq 0.37$ $f(2) = (2)e^{-2} \simeq 0.27$

Then the absolute maximum point is $(1,0.37)$ and the absolute minimum point $(0,0)$

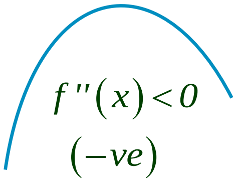
Case (4): Concavity and point of inflection

(1) Concavity

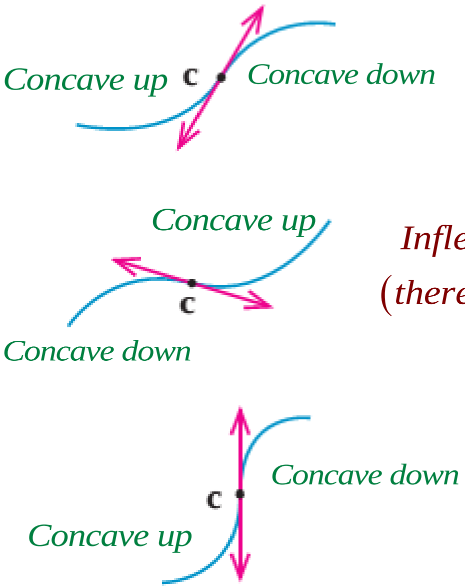
Concave upward "Convex down "



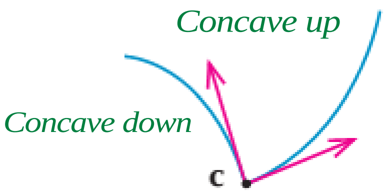
Concave downward "Convex up "



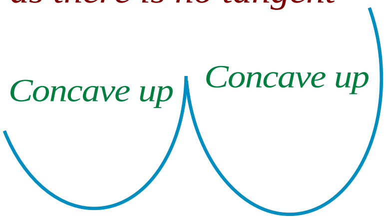
(2) Inflection point



Inflection points
(there is a tangent)



Not an inflection point
as there is no tangent



Not an inflection point
as there is no tangent

Inflection point occurs in one of the following cases

(1) **For polynomial functions**

(i) $f(x)$ is continous , differentiable and $f''(x) = 0$

(ii) The sign of curvature changes from concave up to down or vice versa

(2) **For fractional functions** (derivatives)

(i) $f(x)$ is continous , differentiable and $f''(x) = 0$

(ii) $f'(x)$ doesn't exist except the set of zeroes of the denominator of the original fn

(3) **For double functions**

(i) $f(x)$ must be continous and differentiable $\left[f'(a^-) = f'(a^+) \right]$

(ii) $f'(x) = 0$ or $f'(x)$ doesn't exist

Steps

(1) Find $f''(x)$, then solve $f''(x)=0$.

(2) Determine the sign of $f''(x)$ before and after each point....... if :

 (i) $f''(x)$ changes its sign before and after this point, then this point is an inflection point.

 (ii) $f''(x)$ doesn't change its sign before and after this point, so this point is not an inflection point .

Comparison between Critical and Inflection point

Critical point	Inflection point
Continous	Continous
$\Downarrow$	$\Downarrow$
Not differentiable $f'(a^-)\neq f'(a^+)$	Differentiable $f'(a^-)= f'(a^+)$
$\Downarrow$	$\Downarrow$
$f'(x)=0$	$f''(x)=0$
$\Downarrow$	$\Downarrow$
Curvature changes	Curvature changes

Example (1)

Discuss the concavity of : $f(x)= x^3-6x^2+x+1$

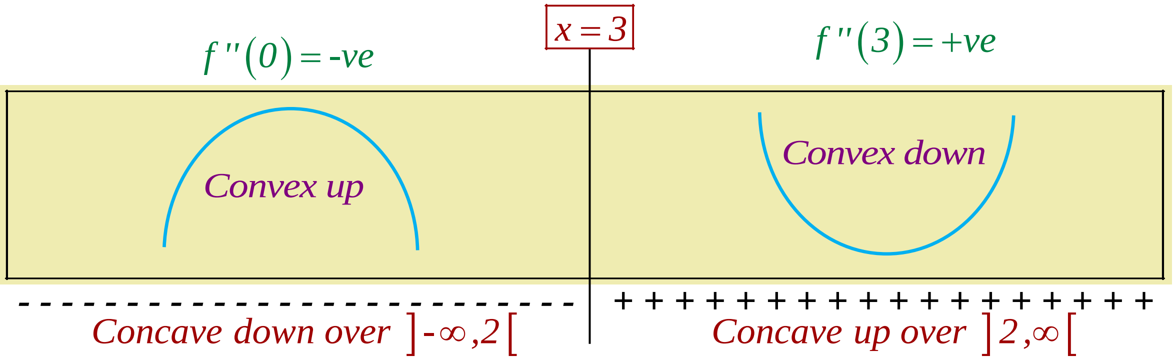
Answer

$\because$ $f(x)$ is continous and differentiable on R

$f'(x)=3x^2-12x+1 \Rightarrow f''(x)=6x-12 \Rightarrow$ for $f''(x)=0 \Rightarrow 6x-12=0 \Rightarrow \boxed{\therefore x=2}$

To check that this point is an inflection or not :

Take point before $\boxed{x=2}$ and another point after $\boxed{x=2}$



So , since there is a **Concave down** then **Concave up** , this means that $x=2$ is an inflection point , then substitute in the original function : $f(2)=(2)^3-6(2)^2+2+1=-13$

Then the inflection point is $(2,-13)$

Example (2)

Consider the function $f(x)=\frac{1}{20}x^5-\frac{1}{12}x^4+x+6$, find the inflection point of $f(x)$ if exists and describe the concavity of f .

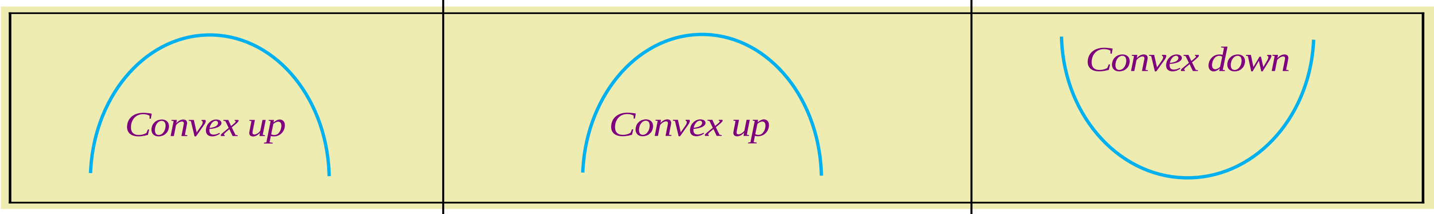
Answer

$\because$ $f(x)$ is continous and differentiable on R

$f'(x)=\frac{1}{4}x^4-\frac{1}{3}x^3+1 \Rightarrow f''(x)=x^3-x^2 \Rightarrow$ for $f''(x)=0 \Rightarrow x^3-x^2=0 \Rightarrow x^2(x-1)=0$

$\boxed{\therefore x=0}$ and $\boxed{\therefore x=1}$

$f''(-1)=-ve$ | $f''(0.5)=-ve$ | $f''(2)=+ve$



- | - | + + + + + + + + + + + + + +

Concave down over $]-\infty,0[$ | Concave down over $]0,1[$ | Concave up over $]1,\infty[$

The only inflection point is at $\boxed{x=1}$: $\therefore f(1)=\frac{1}{20}-\frac{1}{12}+1+6=6.966\simeq 7$

Then the inflection point is $(1,7)$

Example (3)

Determine the intervals over which the curve of the function $f(x)=x^2-2$ is convex upwards and convex downwards and also find the points of inflection if exists .

Answer

$\because$ $f(x)$ is continous and differentiable on R

$f'(x)=2x \Rightarrow f''(x)=2>0 \Rightarrow$ then the function is concave upward (convex down).

And $\because$ the curve does not change its concavity at any point , so there is no inflection points for this function .

Example (4)

Determine the intervals of convexity of the function $f(x)=\begin{cases} x^2-4 & x<-2 \\ x^3-3x+2 & x\geq -2 \end{cases}$, then find the inflection point and the equation of tangent if exists .

Answer

$\because\ f(x)$ is continous at $x=-2$ as $f(-2)=f(-2^-)=f(-2^+)=0$

$\because\ f(x)$ is a double function, we have to get the critical points from:

(i) Check if : $f'(-2^+)=f'(-2^-)$ by using defferentiability $\qquad (ii)\ f''(x)=0$

(i) For $f'(-2^-)$: $\lim\limits_{h\to 0^+}\dfrac{f(-2+h)-f(-2)}{h} \Rightarrow \lim\limits_{h\to 0^+}\dfrac{(-2+h)^2-4-\left[(-2)^2-4\right]}{h}=-4$

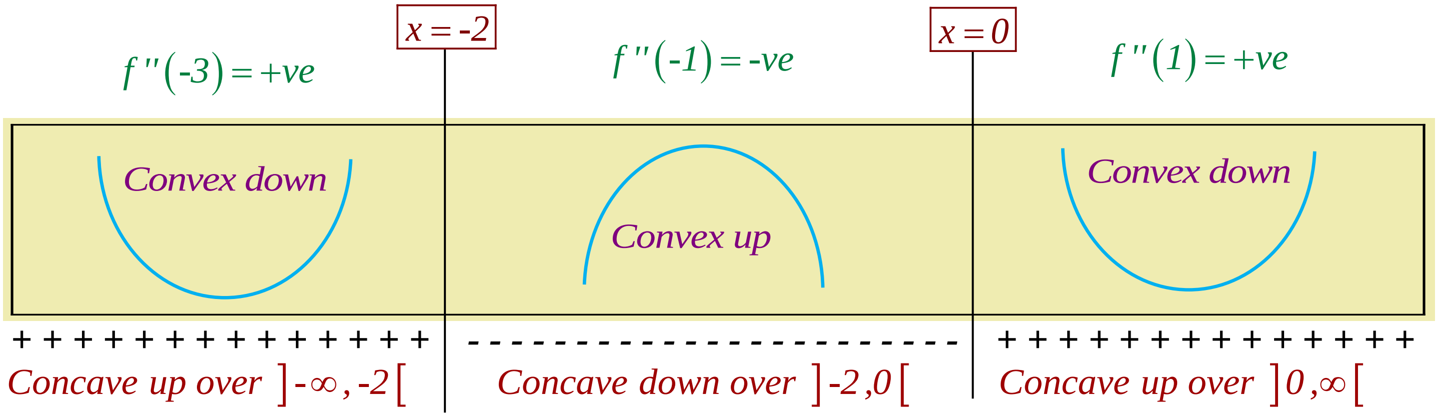
For $f'(-2^+)$: $\lim\limits_{h\to 0^+}\dfrac{(-2+h)^3-3(-2+h)+2-\left[(-2)^3-3(-2)+2\right]}{h}=9$

$\because\ f'(-2^-)\neq f'(-2^+) \Rightarrow$ then $\boxed{x=-2}$ is not an Inflection point

$(ii)\ \because\ f'(x)=\begin{cases} 2x & x<-2 \\ 3x^2-3 & x\geq -2 \end{cases} \Rightarrow f''(x)=\begin{cases} 2 & x<-2 \\ 6x & x\geq -2 \end{cases}$

For $x<-2$: $2=0 \Rightarrow$ Can't be

For $x\geq -2$: $6x=0 \Rightarrow \boxed{\therefore\ x=0\ (agreed)}$ is an inflection point



The only inflection point is at $\boxed{x=0}$: $\therefore\ f(0)=(0)^3-3(0)+2=2$

Then the inflection point is $(0\ ,2)$

Then we can find the equation of tangent at this point

$\because$ The point of tangency is $(0\ ,2)$

And $f'(x)=\begin{cases} 2x & x<-2 \\ 3x^2-3 & x\geq -2 \end{cases} \Rightarrow \therefore$ at $x=0 \Rightarrow \boxed{\dfrac{dy}{dx}=-3}$

Then the equation of tangent line : $y-2=-3(x-0) \Rightarrow \boxed{\therefore\ y+3x-2=0}$

Example (5)

If the function $f(x)=\begin{cases} 2x^2+ax+b & x\geq 1 \\ 3x-x^2 & x<1 \end{cases}$ is differentiable on R

(1) Find the constants a and b

(2) Determine the intervals of convexity up and convexity down .

(3) Find the inflection point and the equation of tangent if exist .

Answer

$\because f(x)$ is differentiable at $x=1 \Rightarrow \therefore f(x)$ is continous at $x=1$

$\therefore f(1)=f(1^+)=2+a+b---(1)$ and $f(1^-)=\lim\limits_{x\to 1^-} 3x-x^2=2---(2)$

$\therefore$ From (1) and (2): $2+a+b=2 \Rightarrow \boxed{\therefore a+b=0---(3)}$

$\because f(x)$ is a double function, we have to get the critical points from:

$\because f(x)$ is differentiable at $x=1 \Rightarrow f'(1^+)=f'(1^-)$

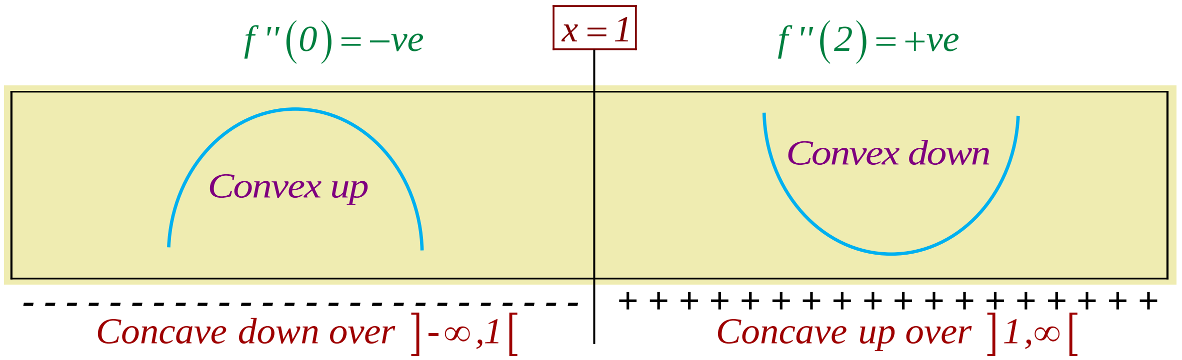
$$(i) \text{For } f'(1^-): \lim_{h\to 0^+}\frac{f(1+h)-f(1)}{h} \Rightarrow \lim_{h\to 0^+}\frac{3(1+h)-(1+h)^2-\left[3(1)-(1)^2\right]}{h}=1$$

$$\text{For } f'(1^+): \lim_{h\to 0^+}\frac{2(1+h)^2+a(1+h)+b-[2+a+b]}{h}=4+a$$

$\therefore 4+a=1 \Rightarrow$ then $\boxed{a=-3} \Rightarrow$ from (3) $\boxed{\therefore b=3} \Rightarrow \therefore f(x)=\begin{cases} 2x^2-3x+3 & x\geq 1 \\ 3x-x^2 & x<1 \end{cases}$

$(ii) \because f'(x)=\begin{cases} 4x-3 & x\geq 1 \\ 3-2x & x<1 \end{cases} \Rightarrow f''(x)=\begin{cases} 4 & x\geq 1 \\ -2 & x<1 \end{cases}$

For $x<1$: $-2=0 \Rightarrow$ Can't be For $x\geq 1$: $4=0 \Rightarrow$ Can't be



The only inflection point is at $\boxed{x=1}$: $\therefore f(1)=2 \Rightarrow \therefore$ the inflection point is $(1,2)$

Then we can find the equation of tangent at this point

$\because$ The point of tangency is $(1,2)$

And $f'(x)=\begin{cases} 4x-3 & x\geq 1 \\ 3-2x & x<1 \end{cases} \Rightarrow \therefore$ at $1=0 \Rightarrow \boxed{\dfrac{dy}{dx}=1}$

Then the equation of tangent line : $y-2=(x-1) \Rightarrow \boxed{\therefore y-x-1=0}$