Examples

Example (1)

The side of a cube increases at the rate of $0.02\ cm/sec$, while its surface area increases at an instant at the rate of $0.08\ cm^2/sec$, find the length of the cube side at this instant and also find the rate of increase of its volume .

Answer

Let " x " is the length of the cube side , "S" is its surface area , "V" is its volume .

Where $\frac{dx}{dt} = 0.02$ And $\frac{ds}{dt} = 0.08$

So $\because S = 6x^2 \Rightarrow \therefore \frac{ds}{dt} = 12x\frac{dx}{dt} \Rightarrow \therefore 0.08 = 12x \times 0.02 \Rightarrow \boxed{\therefore x = \frac{1}{3}\ cm}$

And $\because V = x^3 \Rightarrow \therefore \frac{dv}{dt} = 3x^2\frac{dx}{dt} \Rightarrow \therefore \frac{dv}{dt} = 3 \times \frac{1}{9} \times 0.02 = \frac{1}{150}\ cm^3/sec$.

Example (2)

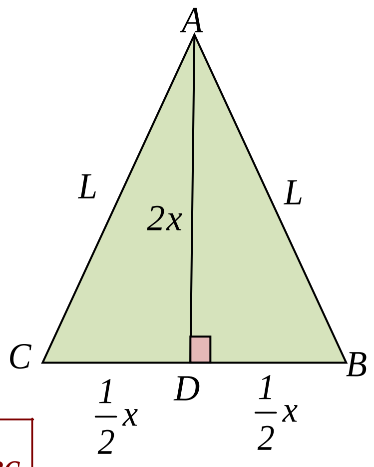
A lamina in the shape of an isosceles triangle, whose height length is twice the length of its base, if its base increases at the rate of $0.04\ cm/sec$. then find :

(a) The increasing rate in its surface area , when its base length is 8 cm .

(b) The rate of change in the length of each of its two equal sides .

Answer

Let the base of the triangle be of length x cm , its height is 2x cm, its surface area is $S\ cm^2$, and the length of each equal sides is L .



where $\frac{dx}{dt} = 0.04\ cm/sec.$

(a) We want to get $\frac{ds}{dt}$ at $x = 8$

The relation between S and X is $S = \frac{1}{2} \times x \times 2x \Rightarrow \therefore S = x^2$

Differentiating with respect to time (t),

$\therefore \frac{ds}{dt} = 2x\frac{dx}{dt} \Rightarrow \therefore \frac{ds}{dt} = 2 \times 8 \times 0.04 \Rightarrow \boxed{\therefore \frac{ds}{dt} = 0.64\ cm^2/sec.}$

(b) We want to get $\frac{dL}{dt}$:

The relation between L and X is : $(AB)^2 = (AD)^2 + (DB)^2$

$\therefore L^2 = (2x)^2 + \left(\frac{1}{2}x\right)^2 \Rightarrow \therefore L^2 = \frac{17}{4}x^2 \Rightarrow \therefore L = \frac{\sqrt{17}}{2}x$

Differentiating with respect to time (t) we get :

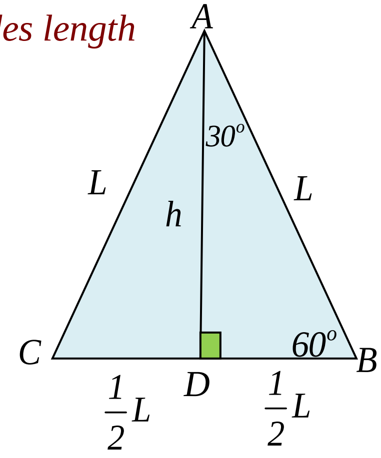
$\frac{dL}{dt} = \frac{\sqrt{17}}{2}\frac{dx}{dt} \Rightarrow \therefore \frac{dL}{dt} = \frac{\sqrt{17}}{2} \times 0.04 \Rightarrow \boxed{\therefore \frac{dL}{dt} = 0.02\sqrt{17}\ cm/sec.}$

Example (3)

A lamina in the shape of an equilateral triangle, if its surface area changes at the rate of 0.6 cm^2 / min, find the rate of change of the length of its side when its height is equal to $6\sqrt{3}$ cm .

Answer

Let "L" is the length of the side , "h" its height and "S" is its surface area .



$\because \frac{ds}{dt} = 0.6$, we have to get a relation between the height and the sides length

$\because \sin 60^o = \frac{h}{L} \Rightarrow h = L \sin 60^o \Rightarrow \boxed{\therefore h = \frac{\sqrt{3}}{2} L}$

$\because S = \frac{\sqrt{3}}{4} L^2 \Rightarrow \therefore \frac{ds}{dt} = \frac{2\sqrt{3}}{4} \times L \times \frac{dL}{dt}$ " so we have to get L "

When $h = 6\sqrt{3}$ cm $\Rightarrow \therefore 6\sqrt{3} = \frac{\sqrt{3}}{2} L \Rightarrow \boxed{\therefore L = 12 \text{ cm}}$

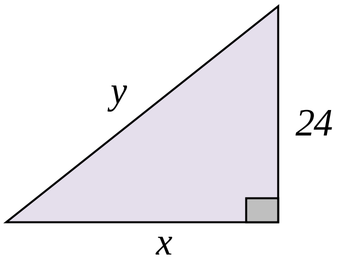
$\therefore 0.6 = \frac{\sqrt{3}}{2} \times 12 \times \frac{dL}{dt} \Rightarrow \boxed{\therefore \frac{dL}{dt} = \frac{0.6}{6\sqrt{3}} = \frac{\sqrt{3}}{30} \text{ cm / min}}$

Example (4)

A man walks at the rate of 6 m / sec , towards the base of a tower of height 24 metres . At what rate does the man approach the top of the tower when he is at a distance of 32 metres from the base of the tower ?

Answer

Let " x " is the distance from the base and , " y " is the distance from the top .



$\frac{dx}{dt} = -6$ m / sec (as the distance to the base decreases) & $\frac{dy}{dt} = ??$ at $x = 32$ m

$\because y^2 = x^2 + (24)^2$

$\therefore 2y\frac{dy}{dt} = 2x\frac{dx}{dt} \Rightarrow \therefore y\frac{dy}{dt} = x\frac{dx}{dt} \Rightarrow \therefore \frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$ "we have to get y"

when $x = 32$ m $\Rightarrow y = \sqrt{(24)^2 + (32)^2} = 40$ m

$\therefore 40\frac{dy}{dt} = 32 \times (-6) \Rightarrow \boxed{\therefore \frac{dy}{dt} = \frac{-24}{5} = -4.8 \text{ m / sec .}}$

Example (5)

A point (x, y) moves along the curve: $x^2-3xy+y^2-5x+4y+3=0$, if the rate of change of its x - axis with respect to the time (t) equals 1 at the point $(2,-1)$, then calculate the rate of change of its Y - axis with respect to the time (t) at the same point .

Answer

By differentiating with respect to time we get :

$$2x\frac{dx}{dt}-3x\frac{dy}{dt}-3y\frac{dx}{dt}+2y\frac{dy}{dt}-5\frac{dx}{dt}+4\frac{dy}{dt}=0$$

$$\therefore\ (2x-3y-5)\frac{dx}{dt}+(-3x+2y+4)\frac{dy}{dt}=0$$

$$\text{When } \frac{dx}{dt}=1 \text{ at } (2,-1) \Rightarrow x=2\ ,\ y=-1$$

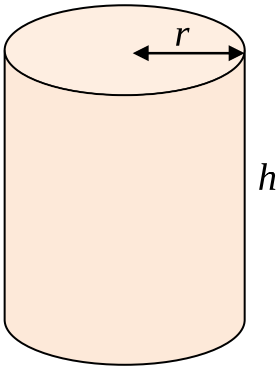
$$\therefore\ (4+3-5)\times 1+(-6-2+4)\ \frac{dy}{dt}\ =0 \Rightarrow \therefore\ 2-4\frac{dy}{dt}\ 0 \quad \Rightarrow \quad \therefore\ \frac{dy}{dt}\ =\ \frac{1}{2}$$

Example (6)

The height of a cylinder equals $\frac{7}{6}$ the length of the diameter of its base, find the rate of change of its volume when the length of its diameter is equal to 12 cm, and the rate of change of its height is 0.01 cm / sec.

Answer

Let r denotes the radius of the base and h is its height



$$\because\ h=\frac{7}{6}D \Rightarrow \therefore\ \frac{dh}{dt}=0.01\ cm/s \quad \Rightarrow \quad \frac{dv}{dt}=??$$

$$\because \text{ The volume of the cylinder } (V)=\pi\ r^2h=\pi\left(\frac{D}{2}\right)^2 h=\frac{\pi}{4}\ D^2h$$

$$\text{And } \because\ h=\frac{7}{6}D \Rightarrow \therefore\ D=\frac{6}{7}h \Rightarrow \therefore\ V=\frac{\pi}{4}\left(\frac{6}{7}h\right)^2 h=\frac{\pi}{4}\times\frac{36}{49}h^2\times h$$

$$\therefore\ V=\frac{9\pi}{49}\times h^3 \Rightarrow \boxed{\therefore\ \frac{dv}{dt}=\frac{9\times 3\pi}{49}\times h^2\times\frac{dh}{dt}---(1)} \text{ "we have to get } h^2\text{"}$$

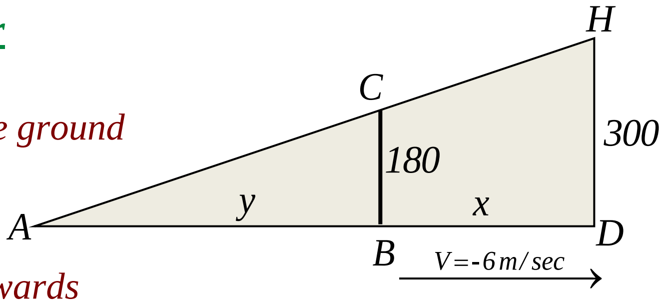
$$\text{When } D=12\ cm \Rightarrow h=\frac{7}{6}\times 12=14\ cm$$

$$\text{Substitute in } (1): \boxed{\therefore\ \frac{dv}{dt}=\frac{27\pi}{49}\times(14)^2\times(0.01)=\frac{27}{25}\pi}$$

Example (7)

A man of height 180 cm walks with a velocity 6 m / sec. towards a light pole with a lamp hanged at its top and of a height 3 metres. Find the rate of change of the length of the man's shadow on the ground . And find the velocity of the end point of the man's shadow .

Answer



Let the man be at a distance x from the light pole

And y is the length of the shadow of the man on the ground

Velocity is always $\frac{distance}{time} = \frac{dx}{dt} = -6\ m/sec$

Note: The sign $(-)$ is because the man is going towards

The pole lamp so the distance here is decreasing according to time

In $\Delta\ AHD$: $\because \Delta\ ABC \sim \Delta\ ADH \Rightarrow \therefore \frac{AB}{AD} = \frac{BC}{BH} \Rightarrow \frac{y}{y+x} = \frac{180}{300} = \frac{3}{5}$

$\therefore 3x + 3y = 5y \Rightarrow \therefore 2y = 3x$ and by differentiating with respect to time

We get : $2\frac{dy}{dt} = 3\frac{dx}{dt}$ but $\frac{dx}{dt} = -6\ m/sec.$ $\Rightarrow \therefore 2\frac{dy}{dt} = 3(-6) = -18 \Rightarrow \boxed{\therefore \frac{dy}{dt} = -9\ m/sec}$

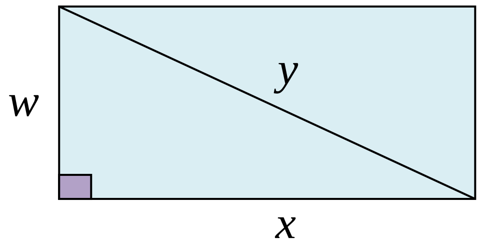
Let the end point of the man's shadow be $Z = x + y \Rightarrow \therefore \frac{dz}{dt} = \frac{dx}{dt} + \frac{dy}{dt} = -6 - 9 = -15\ m/sec$

Example (8)

A thin mettalic lamina in the form of a rectangle, its length is $\frac{4}{5}$ its diagonal, it shrinks uniformly by cooling keeping its geometric form and ratio between its dimension .

At a moment its diameter shrinks with the rate 2.5 cm / min . and at the same moment its surface area shrinks with rate 60 cm^2 / min . Find the surface area of the lamina at this moment .

Answer



$x = \frac{4}{5}y$, $\frac{dy}{dt} = -2.5\ cm/min$, $\frac{ds}{dt} = -60\ cm^2/\ min$.

$\boxed{\because \text{Surface Area } (S) = w \times x\ ---(1)}$

$\because\ y^2 = w^2 + x^2 \Rightarrow y^2 = w^2 + \frac{16}{25}y$

$w^2 = y^2 - \frac{16}{25}y^2 \Rightarrow w^2 = \frac{9}{25}y^2$

$\therefore\ w = \frac{3}{5}y \Rightarrow \therefore\ S = \frac{3}{5}y \times \frac{4}{5}y = \frac{12}{25}y^2$

$\frac{ds}{dt} = \frac{24}{25}y\frac{dy}{dt} \Rightarrow -60 = \frac{24}{25}y \times (-2.5) \Rightarrow \boxed{\therefore\ y = 25\ cm}$

Then $\boxed{x = \frac{4}{5}(25) = 20\ cm}$ and $\boxed{w = \frac{3}{5}(25) = 15\ cm}$

$\therefore$ Surface area of lamina $= 15 \times 20 = 300\ cm^2$

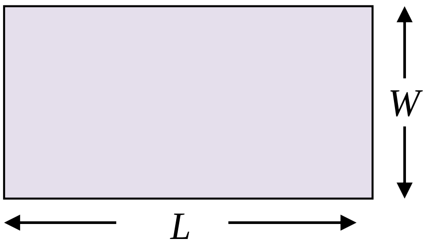
Example (9)

A piece of metal in the shape of a rectangle. If its length decreases at the rate of 0.3 cm / sec . while its width increases at the rate of 0.1 cm / sec, calculate the rate of change of its surface area when its length is equal to 8 cm, and its width is 6 cm .

Answer

Let "L" is the Length and " W " is the width and " S " is its surface area .

$\because \frac{dL}{dt} = -0.3$ And $\frac{dw}{dt} = 0.1$



And $\because S = L\,W$

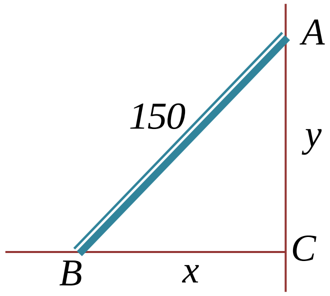
$\therefore \frac{ds}{dt} = L\frac{dw}{dt} + w\frac{dL}{dt}$

$\therefore \frac{ds}{dt} = 8\times 0.1 - 6\times 0.3 = 0.8 - 1.8 = -1\ cm^2/sec$.

Example (10)

A ladder $\overline{AB}$ of length 150 cm rests with its upper end A against a vertical wall , and with its lower end B on a horizontal ground . If its lower end B slides away from the wall at the rate 20 cm / sec . Find the velocity of sliding of its upper end on the wall , when its lower end B is 90 cm distance from the wall, then find the distance of A from the ground when the speed of A equals twice the speed of B .

Answer



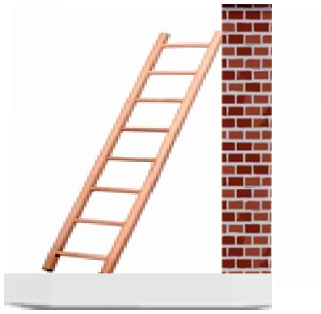
$\frac{dx}{dt} = 20\ cm/sec$ & $\frac{dy}{dt} = ???$ when $x = 90\ cm$

$\boxed{\because x^2 + y^2 = (150)^2 = 22500 \;---(1)}$

And $\because 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \;(\div 2) \Rightarrow \boxed{\therefore x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \;---(2)}$

When $x = 90$: from $(1) \Rightarrow \therefore y^2 = 14400 \Rightarrow \boxed{\therefore y = 120\ cm}$

From (2) : $\therefore 90 \times 20 + 120\frac{dy}{dt} = 0 \Rightarrow \boxed{\therefore \frac{dy}{dt} = -15\ cm/sec\,.}$



When $\frac{dy}{dt} = -2\frac{dx}{dt}$ & $y = ????$

From (2): $\because x\frac{dx}{dt} + y\frac{dy}{dt} = 0 \Rightarrow \therefore x\frac{dx}{dt} - 2y\frac{dx}{dt} = 0 \Rightarrow \therefore x - 2y = 0$

$\therefore x = 2y$, from (1) : $4y^2 + y^2 = 22500 \Rightarrow 5y^2 = 22500 \Rightarrow \therefore y^2 = \frac{22500}{5}$

$$\boxed{\therefore y = \frac{150}{\sqrt{5}} = 30\sqrt{5}\ cm}$$

Example (11)

The volume of a spherical rubber ballon is $56\,\pi\ cm^3$, filled with gas, and as a result of the leakage of the gas , the volume of the balloon decreases at a rate of $4\,\pi\ cm^3 / min$, keeping its spherical shape . Find :

(1) *The rate of change of the length of the radius of the balloon, when the length of the radius equals 2 cm .*

(2) *The rate of change of the length of the radius of the balloon 5 minutes after the gas begins to leak .*

Answer

(1) ....

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$$\boxed{Ans:\ \frac{dr}{dt} = -\frac{1}{4}\ cm/min}$$

(2) After 5 minutes magnitude of leakage $= 5 \times 4\pi = 20\pi\ cm^3$

$\therefore$ The volume will be $= 56\pi - 20\pi = 36\pi\ cm^3 \Rightarrow \therefore\ 36\pi = \frac{4}{3}\pi r^3 \Rightarrow \therefore\ r^3 = 27 \Rightarrow \boxed{\therefore\ r = 3}$

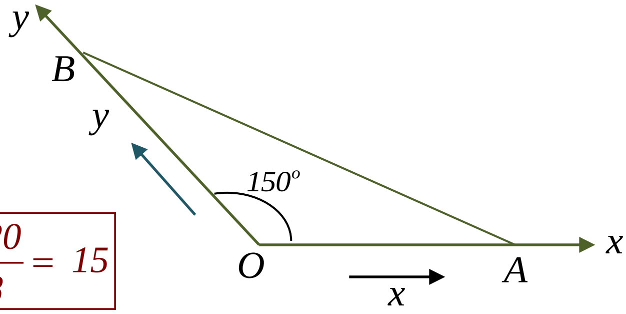
$\because \frac{dv}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow \therefore\ -4\pi = 4\pi \times (3)^2 \frac{dr}{dt} \Rightarrow \boxed{\therefore\ \frac{dr}{dt} = -\frac{1}{9}\ cm/min}$

Example (12)

$\overrightarrow{OX}$ and $\overrightarrow{OY}$ are two rays where $m(\angle XOY) = 150^o$, a point (A) moves along $\overrightarrow{OX}$ with uniform velocity 4 cm / sec while another point (B) moves along $\overrightarrow{OY}$ such that the surface area of $\Delta\ OAB$ is $30\ cm^2$. Find the magnitude and direction of the velocity of point (B) when the length of $\overline{OA}$ equals 8 cm .

Answer

$\frac{dx}{dt} = 4\ cm/sec$ and area of $\Delta\ OAB = 30\ cm^2$ when $x = 8$



$\because$ Area of $\Delta\ OAB = \frac{1}{2}\, x\, y\, Sin 150^o = \frac{1}{4}\, x\, y$

$\therefore\ \frac{1}{4}\, y\, x = 30 \Rightarrow \boxed{\therefore\ x\, y = 120 --- (1)}$

$\therefore\ x\frac{dy}{dt} + y\frac{dx}{dt} = 0$, so when $x = 8 \Rightarrow \boxed{\therefore\ y = \frac{120}{8} = 15}$

$\therefore\ 8\frac{dy}{dt} + 15 \times 4 = 0 \Rightarrow \boxed{\therefore\ \frac{dy}{dt} = -7.5\ cm/sec}$

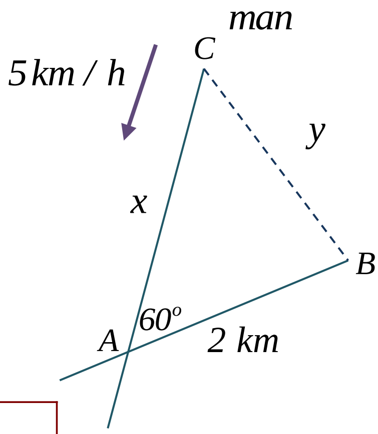
Example (13)

Two roads intersect at A, there is a house B lies on one of the two roads such that $AB = 2\,km$ A man walks in the other road with a velocity of $5\ km/hr$ towards A, find the rate of change of the distance between the man and the house at the instant when he is at 1.5 km, far away from A : (a) If the measure of the angle between the two roads is 60^o .

(b) If the two roads are perpendicular .

Answer

Let " x " is the distance between the man C and A and y is the distance between the man C and the house B and $\frac{dx}{dt} = -5\ km/h$



(a) $\because\ y^2 = x^2 + (2)^2 - 2x \times 2\ Cos\ 60°$

$\therefore\ y^2 = x^2 - 2x + 4 \Rightarrow$ so $\boxed{2y\frac{dy}{dt} = (2x-2)\frac{dx}{dt} - - - (1)}$

When $x = 1.5 \Rightarrow \therefore\ y^2 = 2.25 - 3 + 4 \Rightarrow \boxed{\therefore\ y = \frac{1}{2}\sqrt{13}}$

From (1) : $\therefore\ \sqrt{13}\frac{dy}{dt} = (3-2)(-5) \Rightarrow \boxed{\therefore\ \frac{dy}{dt} = \frac{-5\sqrt{13}}{13}\ km/hr}$

(b) ....

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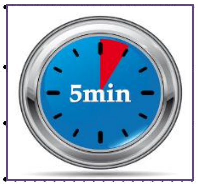

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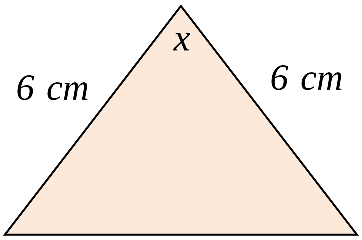
.... $\boxed{Ans : \frac{dy}{dt} = -3\ km/hr}$

Example (14)

The length of each of the two sides of an isosceles triangle is 6 cm and the measure of the angle between them is x , if x changes at the rate of $\frac{\pi}{90}$ rad. / min . $(\pi = 3.14)$, find the rate of change of the area of the triangle when $x = 30^o$.

Answer

$\frac{dx}{dt} = \frac{\pi}{90}$ rad / min and $\because\ A = \frac{1}{2}(6)^2 Sin\ x$ (x must be in degree)



$\therefore\ A = 18\ Sin\ x \Rightarrow \therefore\ \frac{dA}{dt} = 18\ Cos\ x\ \frac{dx}{dt}$

When $x = 30^o$ and $\frac{dx}{dt} = \frac{\pi}{90}$

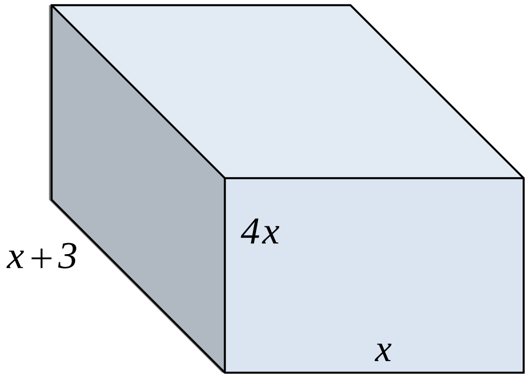
$\therefore\ \frac{dA}{dt} = 18\ Cos\ 30 \times \frac{\pi}{90} = 18 \times \frac{\sqrt{3}}{2} \times \frac{3.14}{90} \Rightarrow \therefore\ \frac{dA}{dt} = 0.54386\ cm^2/min$

Example (15)

A piece of metal in the shape of rectangular box , the length of its base is 3 cm longer than its width and its altitude is 4 times its width , if the piece is heated and expands keeping the same ratio between its dimensions , and the rate of change in volume is 3 cm^3 / min , the width is increasing by $\frac{1}{32}$ cm/min , find the dimensions of the bow at this instant .

Answer

Let the width of the box be x , its length be $x+3$, its altitude be $4x$



$\because \frac{dv}{dt} = 3\ cm^3 / min \quad and \quad \frac{dx}{dt} = \frac{1}{32}\ cm / min$

$\because V = 4x(x+3)(x) = 4x^3 + 12x^2$

$\therefore \frac{dv}{dt} = 12x^2 \frac{dx}{dt} + 24x \frac{dx}{dt}$

At $\frac{dv}{dt} = 3\ cm^3 / min \quad and \quad \frac{dx}{dt} = \frac{1}{32}\ cm / min$

$\therefore 3 = 12x^2\left(\frac{1}{32}\right) + 24x\left(\frac{1}{32}\right) \Rightarrow 3 = \frac{3}{8}x^2 + \frac{3}{4}x$ "multiply by 8" $\Rightarrow \therefore 3x^2 + 6x = 24$

$\therefore x^2 + 2x - 8 = 0 \Rightarrow (x+4)(x-2) = 0 \Rightarrow \boxed{\therefore x = 2}$

Then : width = 2 cm the length is 5 cm the hight is 8 cm

Example (16)

A point (x,y) moves along the curve $y = 4x - 2x^3$ such that its Y – coordinate decreases at the rate of 5 units / sec , find the rate of change of the slope of the tangent to the curve when $x = 2$

Answer

Curve w.r.t. time

$\frac{dy}{dt} = 4\frac{dx}{dt} - 6x^2\frac{dx}{dt} = (4 - 6x^2)\frac{dx}{dt}$

at $x = 2$ and $\frac{dy}{dt} = -5$

$(-5) = (4 - 6(2)^2)\frac{dx}{dt}$

$\boxed{\frac{dx}{dt} = \frac{-5}{-20} = \frac{1}{4}\ unit / sec}$

Slope w.r.t. time

$Slope = S = \frac{dy}{dx} = 4 - 6x^2$

$\therefore \frac{ds}{dt} = -12x\frac{dx}{dt}$

$\therefore$ at $x = 2 \Rightarrow \frac{dx}{dt} = \frac{1}{4}$

$\therefore \frac{ds}{dt} = -12 \times 2 \times \frac{1}{4} = -6$

$\therefore$ The rate of change of the slope of the tangent when $x = 2$ is -6 units / sec .

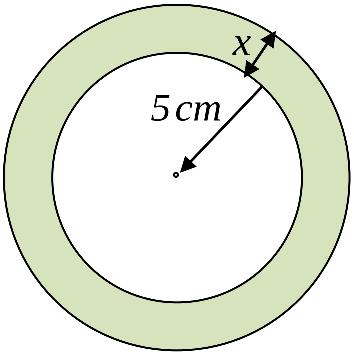
Example (17)

An iron sphere of diameter 10 cm is covered by a uniform layer of ice . If the ice is melted at a rate of $5\ cm^3/min$, find the velocity with which the ice thickness is decreasing when its thickness equals one cm , and find the rate of decrease of the area of the outer surface of the ice layer at this instant .

Answer

Let the thickness of the layer of ice $= x$ then $\frac{dx}{dt} = ??$ when $x = 1$ and $\frac{dv}{dt} = -5\ cm^3/min$

And $V(\text{volume of the layer}) = \frac{4}{3}\pi(5+x)^3 - \frac{4}{3}\pi\times(5)^3$



$\therefore \frac{dv}{dt} = 4\pi(5+x)^2\frac{dx}{dt}$

When $x = 1\ cm$ and $\frac{dv}{dt} = 5\ cm^3/sec$

$\therefore -5 = 4\pi(6)^2\frac{dx}{dt} \Rightarrow \boxed{\therefore \frac{dx}{dt} = \frac{-5}{144\pi}\ cm/sec}$

Let "S" is the surface area of the layer

$\therefore S = 4\pi(5+x)^2 - 4\pi\times(5)^2 \Rightarrow \therefore \frac{ds}{dt} = 8\pi(5+x)\frac{dx}{dt}$

$$\boxed{\therefore \frac{ds}{dt} = 8\pi\times6\times\frac{-5}{144\pi} = \frac{-5}{3}\ cm^2/sec}$$

Example (18)

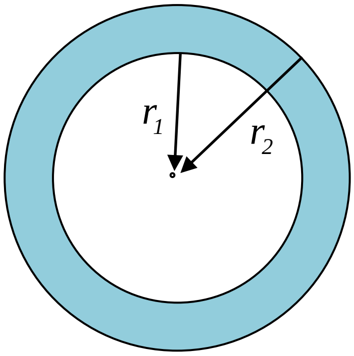
Ahollow sphere has an internal radius r_1 and outside radius r_2 at any instant of time , if r_1 increases at the rate of 1 cm / sec , calculate the rate of change of r_2 , such that the volume of the material of the sphere remains constant at the instant when $r_1 = 3\ cm$ & $r_2 = 9\ cm$.

Answer

The volume of the sphere remains constant means $\frac{dv}{dt} = 0$, $\frac{dr_1}{dt} = 1$

$\because V = \frac{4}{3}\pi r_2^3 - \frac{4}{3}\pi r_1^3 \Rightarrow \therefore \frac{dv}{dt} = 4\pi r_2^2\frac{dr_2}{dt} - 4\pi r_1^2\frac{dr_1}{dt}$

$0 = 4\pi r_2^2\frac{dr_2}{dt} - 4\pi r_1^2\frac{dr_1}{dt}\quad(\div 4\pi) \Rightarrow \therefore r_2^2\frac{dr_2}{dt} - r_1^2\frac{dr_1}{dt} = 0$



At $r_1 = 3\ cm$, $r_2 = 9\ cm$, $\frac{dr_1}{dt} = 1$

$\therefore 81\frac{dr_2}{dt} - 9\times1 = 0 \Rightarrow \boxed{\therefore \frac{dr_2}{dt} = \frac{9}{81} = \frac{1}{9}\ cm/sec}$

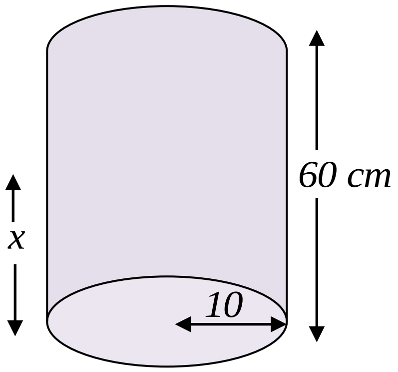
Example (19)

Water is poured into an empty cylinderical container of radius 10 cm , and height 60 cm at the rate of $30\,\pi\ cm^3 / sec$. Find the rate at which the height of the water rises in the container , When does the container become full ?

Answer

Let x be the height of the cylinder moves $\Rightarrow$ then $\frac{dx}{dt} \Rightarrow$ change in hight

In this problem : the raduis is constant and the hight is variable



$\because\ V = \pi r^2 h \Rightarrow V = \pi r^2 x$

Where $r = 10$ cm $\Rightarrow V = 100\,\pi\,x$

$\frac{dv}{dt} = 100\,\pi \frac{dx}{dt} \Rightarrow 30\,\pi = 100\,\pi \frac{dx}{dt} \Rightarrow \boxed{\therefore \frac{dx}{dt} = \frac{3}{10}\ cm/sec}$

This means that for every sec , the height of the water rises 0.3 cm

So , the container becomes full when the height of the water becomes 60 cm .

Or we can say : Every 1 sec $\rightarrow$ 0.3 cm .
Then T $\rightarrow$ 60 cm . $\Rightarrow \boxed{\therefore T = \frac{60 \times 1}{0.3} = 200\ sec}$

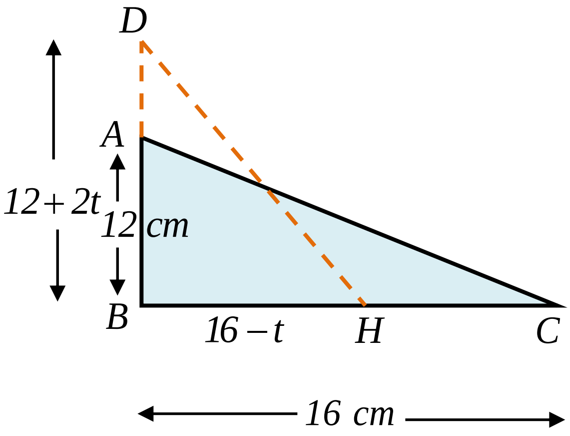
Function of time problems

Example (20)

In an instant, the lengths of the two sides of a right - angled triangle are 12 cm and 16 cm . If the length of the first side increases at the rate of 2 cm / sec and the length of the second side decreases at the rate of 1 cm / sec , find the rate of change of its surface area after 2 seconds from the fixed instant .

Answer

Very important note : when the problem discusses a period of time , we must write the relation in the function of time



$\because$ The length of $(\overline{AB})$ increases at the rate of 2 cm / sec.

Then after t seconds ,its length becomes $(12 + 2t)$ cm .

$\therefore BD = (12 + 2t)$ cm .

$\because$ The length of $(\overline{BC})$ decreases at the rate of 1 cm / sec.

Then after t seconds , its length becomes $(16 - t)$ cm .

$\therefore BH = (16 - t)$ cm .

$\therefore$ The surface area of $\triangle DBH = \frac{1}{2}(12 + 2t)(16 - t) \Rightarrow \therefore S = (6 + t)(16 - t)$

$\therefore S = 96 + 10t - t^2 \Rightarrow \therefore \frac{ds}{dt} = 10 - 2t$

$\therefore$ After 2 seconds we get $\frac{ds}{dt} = 10 - 2 \times 2 \Rightarrow \boxed{\therefore \frac{ds}{dt} = 6\ cm^2 / sec}$

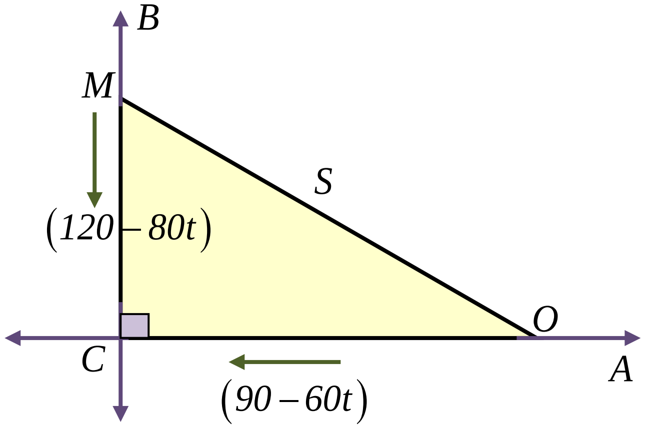
Example (21)

$\overrightarrow{AC}$ and $\overrightarrow{BC}$ are two orthogonal roads, $AC = 90$ km, And $BC = 120$ km, a car is moving from A towards C with a uniform velocity of 60 km / hr, and at the same moment another car was moving from B towards C with a uniform velocity of 80 km / hr, find the rate of change of the distance between the two cars after one hour from the instant they were moving at A and B.

Answer

Very important note : (1) when the problem discusses a period of time, we must write the relation in the function of time

(2) $\because$ The car and the train are moving **towards** from each other $\Rightarrow$ Then we use Subtraction



$$\because\ S^2 = (90-60t)^2 + (120-80t)^2 \quad ---(1)$$

$$\therefore\ 2S\frac{dS}{dt} = 2(90-60t)(-60) + 2(120-80t)(-80)$$

$$\therefore\ S\frac{dS}{dt} = -60(90-60t) - 80(120-80t)$$

After 1 hour :

$$S\frac{dS}{dt} = -60(90-60(1)) - 80(120-80(1)) = -5000$$

$$\therefore \frac{dS}{dt} = \frac{-5000}{S} \quad \text{" so we have to get } S \text{ at } t=1\text{"}$$

From (1): $S^2 = (90-60(1))^2 + (120-80(1))^2 = 2500 \Rightarrow \therefore\ S = 50$

$$\therefore \frac{dS}{dt} = \frac{-5000}{50} = -100 \text{ km/h}$$

Example (22)

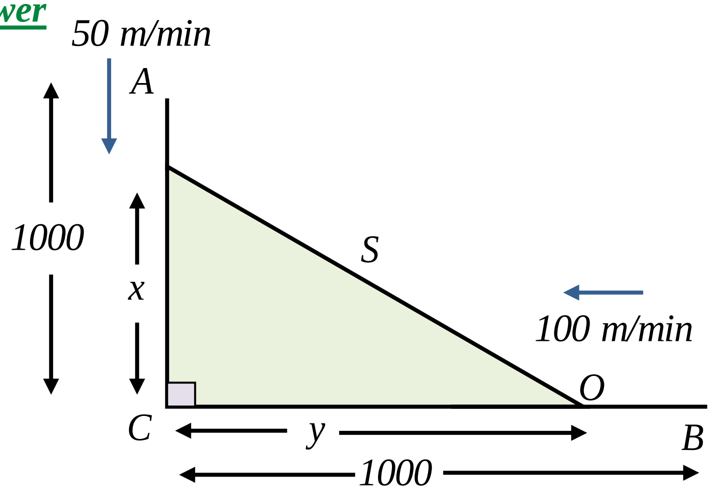
$\overrightarrow{AC}$ and $\overrightarrow{BC}$ are two perpendicular roads where $AC = BC = 1000\ m$, two men started to walk the first from A towards C with a uniform velocity of $50\ m/min$, while the other from B towards C with a uniform velocity of $100\ m/min$, prove that, the distance betweeen the two men after T minutes from moment they started both to walk is given by the relation : $S^2 = (1000 - 50t)^2 + (1000 - 100t)^2$

Then find :

(1) The rate change of the distance between the two men with respect to the time When $t = 4$

(2) When will the two men start to move away from each other .

Answer



∗ After t seconds :

• The man at A walkes $50\,t$,

Then the distance remains till C $(1000 - 50t)$

• Similarly the man at B walks $100\,t$,

Then the distance remains till C $(1000 - 100t)$

So , $\because\ S^2 = x^2 + y^2$

$S^2 = (1000 - 50t)^2 + (1000 - 100t)^2$

(1) Then $2S\dfrac{ds}{dt} = 2(1000 - 50t)(-50) + 2(1000 - 100t)(-100)\quad (\div 2)$

$$S\frac{ds}{dt} = (1000 - 50t)(-50) + (1000 - 100t)(-100)$$

When $t = 4$: $S\dfrac{ds}{dt} = (1000 - 50(4))(-50) + (1000 - 100(4))(-100) = -100000$

$\therefore\ S\dfrac{ds}{dt} = -100000 \Rightarrow \dfrac{ds}{dt} = \dfrac{-100000}{S}$ "we have to get S"

$\therefore$ At $t = 4$: $S = \sqrt{(1000 - 200)^2 + (1000 - 400)^2} = 1000 \Rightarrow \boxed{\therefore\ \dfrac{ds}{dt} = \dfrac{-100000}{1000} = -100\ m/min}$

This means that both men will be close at rate $100\ m/min$.

(2) The two men will be nearer to each other when $\dfrac{ds}{dt} = 0$

$\therefore\ 0 = 2(1000 - 50t)(-50) + 2(1000 - 100t)(-100)$, (divide by -100)

$$1000 - 50t + 2000 - 200t = 0 \Rightarrow 250t = 3000 \Rightarrow \boxed{\therefore\ t = 12\ min}$$

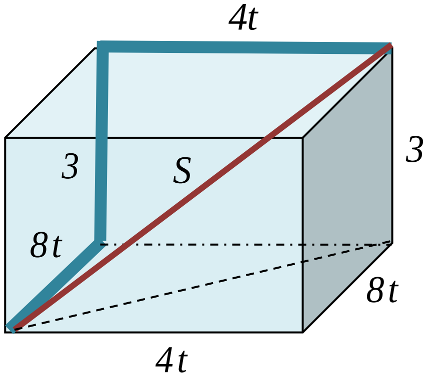
So , after 12 minutes both men will meet each other then both of them will move away from each other when $t > 12$ minutes .

Example (23)

A tunnel for passing trains is under a road for passing cars, the two roads are perpendicular and the shortest distance betwee them is 3 meters. If a car moves with a velocity $4\ m/sec$. and at the same instant a train moves with velocity $8\ m/sec$ from a point being vertically under the car. Find the rate at which they get further from each other after 2 seconds .

Answer

Very important note : when the problem discusses a period of time , we must write the relation in the function of time



$$\because\ S^2=(3)^2+(4t)^2+(8t)^2=80t^2+9---(1)$$

$$\therefore\ 2S\frac{dS}{dt}=160\,t\ \Rightarrow\ \therefore\ \frac{dS}{dt}=\frac{80t}{S}$$

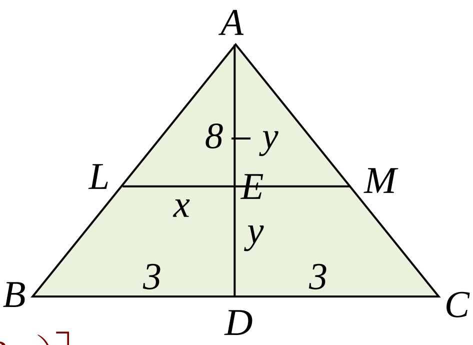
After 2 seconds : $S^2=80(2)^2+9=329\ \Rightarrow\ \therefore\ S=\sqrt{329}\ m$

$$\therefore\ \frac{dS}{dt}=\frac{80(2)}{\sqrt{329}}=\frac{160}{\sqrt{329}}\ \ m/s$$

Example (24)

ABC is an isoseles triangle, the length of its base $\overline{BC}$ is 6 cm, and its height equals 8 cm, $\overline{LM}$ is parallel to $\overline{BC}$ and intersects $\overline{AB}$ at L , $\overline{AC}$ at M . If the height of the trapezium $LBCM$ decreases at a rate $\frac{1}{2}$ cm / sec , find the rate of decreasing of the surface area of the trapezium $LBCM$ when $\overleftrightarrow{LM}$ bisects $\overline{AB}$.

Answer



$\frac{dy}{dt}=-\frac{1}{2}$ $\qquad \frac{ds}{dt}=???$ $\quad$ let $ED=y$ and $AE=8-y$ $\quad LE=x$

$$\because\ \overline{LE}\ //\ \overline{BD}\ \Rightarrow\ \therefore\ \Delta ALE\sim\Delta ABD\ \Rightarrow\ \therefore\ \frac{AL}{AB}=\frac{AE}{AD}=\frac{LE}{BD}$$

$$\therefore\ \frac{x}{3}=\frac{8-y}{8}\ \Rightarrow\ \therefore\ x=\frac{3(8-y)}{8}=3-\frac{3}{8}y$$

$$\because\ \text{Surface area of trapezuim: } S=\frac{1}{2}[6+2x]y=\frac{1}{2}y\left[6+2\left(3-\frac{3}{8}y\right)\right]$$

$$\therefore\ S=6y-\frac{3}{8}y^2\ \Rightarrow\ \therefore\ \frac{ds}{dt}=6\frac{dy}{dt}-\frac{3}{4}y\frac{dy}{dt}$$

When $\overline{LM}$ bisects $\overline{AB}$ $\Rightarrow$ $\therefore\ x=\frac{1}{2}\times3=1.5\ cm$, in this case : $y=4$

$$\therefore\ \frac{dS}{dt}=6\left(-\frac{1}{2}\right)-\frac{3}{4}\times4\left(-\frac{1}{2}\right)=-3+\frac{3}{2}=-\frac{3}{2}\ \ cm^2/s$$

Example (25)

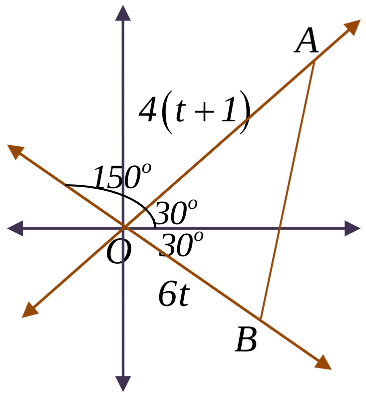
From the origin point (O) , in the perpendicular coordinates plane , a particle A moves in the direction 30^o north of the east with velocity of magnitude 4 meter/min , after one minute another particle B moves from the same origin point (O) along the staright line $\overleftrightarrow{OB}$ whose equation is $x+\sqrt{3}\ y=0$ with velocity of magnitude 6 meter/min in the direction which makes angle BOA acute , find the rate of change of the distance between the two particles A and B two minutes after particle B has started its motion .

Answer

The slope of $\overleftrightarrow{OB}=\frac{-1}{\sqrt{3}}=Tan150^o \Rightarrow$ then $(<AOB)$ is the only acute angle

When B is moving in the direction 30^o south of east

$$\therefore\ m(<AOB)=60^o$$



After (t) minutes from the instant of starting motion of B

Then $OB=6t$ and $OA=4(t+1)$

$$\therefore\ (AB)^2=(OA)^2+(OB)^2-2(OA)(OB)Cos60^o$$

$$\therefore\ (AB)^2=(4t+4)^2+(6t)^2-2(4t+4)(6t)\times\frac{1}{2}$$

$$\boxed{\therefore\ S^2=28t^2+8t+16---(1)} \Rightarrow 2S\frac{ds}{dt}=56t+8 \Rightarrow S\frac{ds}{dt}=28t+4$$

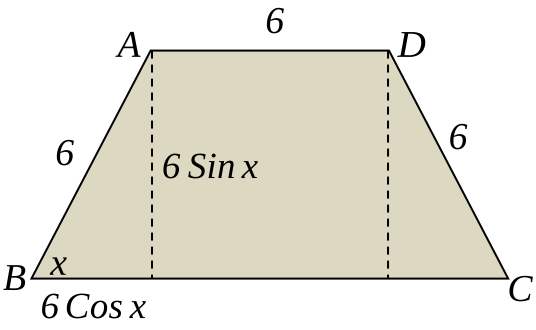
After 2 minutes : $S\frac{ds}{dt}=28(2)+4 \Rightarrow \boxed{\frac{ds}{dt}=\frac{60}{S}}$ "we have to get S"

At $t=2$: $S^2=128(2)^2+8(2)+16=144 \Rightarrow \boxed{S=12\ m} \Rightarrow \therefore\ \frac{ds}{dt}=\frac{60}{12} \Rightarrow \boxed{\frac{ds}{dt}=5\ m/min}$

Example (26)

ABCD is a trapezoid, where $\overline{AD} \parallel \overline{BC}$, $AD = AB = CD = 6$ *cm* , $m(< ABC) = x$, *if this angle is increasing by* 2^o / min , *find the rate of change of the area of the trapezoid when* $x = 30^o$

Answer



$\because$ Area of trapezuim $= \frac{1}{2}(\text{sum of two parallel bases}) \times h$

In ΔABE : $Sin\,x = \frac{AE}{6} \Rightarrow \boxed{AE = 6\,Sin\,x}$

And $Cos\,x = \frac{BE}{6} \Rightarrow \boxed{BE = 6\,Cos\,x}$

$\therefore A = \frac{1}{2}\left[(6Cos\,x + 6 + 6\,Cos\,x) + 6\right](6\,Sin\,x)$

$\therefore A = \frac{1}{2}(12Cos\,x + 12)(6\,Sin\,x) = (6\,Cos\,x + 6)(6\,Sin\,x)$

$\frac{dA}{dt} = (6\,Cos\,x + 6)(6Cos\,x)\frac{dx}{dt} + (6\,Sin\,x)(\text{-}6\,Sin\,x)\frac{dx}{dt}$, then at $x = 30^o$ and $\frac{dx}{dt} = 2^o$

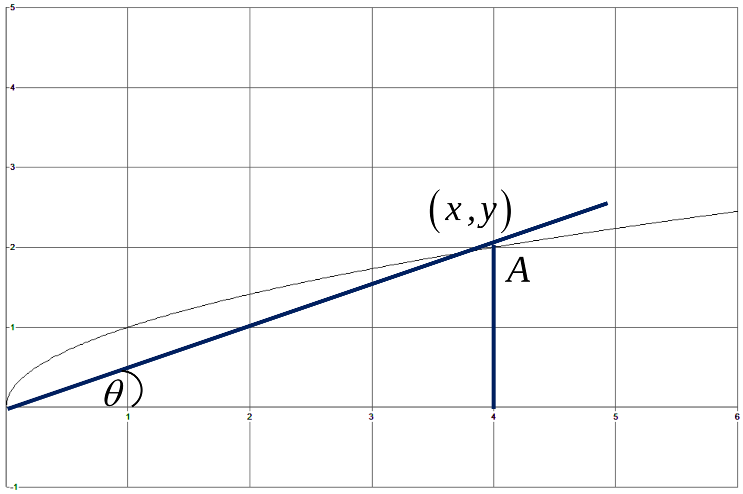
$\therefore \frac{dA}{dt} = \left(6\,Cos\left(30^o\right) + 6\right)\left(6Cos\left(30^o\right)\right)(2) + \left(6\,Sin\left(30^o\right)\right)\left(\text{-}6\,Sin\left(30^o\right)\right)(2) \simeq 98.3\,cm^2 / min$

Example (27)

A particle (x,y) is moving along the curve of the function $y=\sqrt{x}$, when $x=4$, the $y-$component of the position of the particle is increasing at rate 1 cm/sec

(i) What is the rate of change of the $x-$component at this moment?

(ii) What is the rate of change of the distance from the origin to the particle at the same moment



(iii) What is the rate of change of the inclination angle θ at the same moment?

Answer

$(i) \because\ y=x^{\frac{1}{2}} \Rightarrow \therefore \frac{dy}{dt}=\frac{1}{2}x^{\frac{-1}{2}}\frac{dx}{dt} \Rightarrow 1=\frac{1}{2}(4)^{\frac{-1}{2}}\frac{dx}{dt} \Rightarrow \boxed{\therefore \frac{dx}{dt}=4\ cm/sec}$

(ii) $\because$ the distance between the origin $(0,0)$ and the particle (x,y)

is $S=\sqrt{(y-y_1)^2+(x-x_1)^2}=\sqrt{(y-0)^2+(x-0)^2}=\sqrt{y^2+x^2}=\sqrt{x+x^2}=(x^2+x)^{\frac{1}{2}}$

$$\therefore \frac{dS}{dt}=\frac{1}{2}(x^2+x)^{\frac{-1}{2}}(2x+1)\frac{dx}{dt} \Rightarrow \text{at } x=4 \text{ and } \frac{dx}{dt}=4\ cm/sec$$

$$\therefore \frac{dS}{dt}=\frac{1}{2}\left((4)^2+(4)\right)^{\frac{-1}{2}}(2(4)+1)(4)=\frac{18}{\sqrt{20}}=\frac{18}{2\sqrt{5}}=\frac{9\sqrt{5}}{5}\ \text{cm/sec}$$

(iii) $\because$ The rate of change of the angle of inclination θ is $\frac{d\theta}{dt}$

And $\boxed{\because\ Tan\theta=\frac{y}{x}=\frac{\sqrt{x}}{x}=x^{\frac{-1}{2}}---(1)} \Rightarrow \therefore$ by differentiating w.r.t time:

$$\therefore Sec^2\theta\frac{d\theta}{dt}=-\frac{1}{2}x^{\frac{-3}{2}}\frac{dx}{dt} \Rightarrow \boxed{\therefore \frac{d\theta}{dt}=\frac{-\frac{1}{2}x^{\frac{-3}{2}}\frac{dx}{dt}}{Sec^2\theta}=-\frac{1}{2}x^{\frac{-3}{2}}\frac{dx}{dt}Cos^2\theta---(2)}$$

From (1): at $x=4 \Rightarrow Tan\theta=\frac{1}{2} \Rightarrow \boxed{\therefore Cos\theta=\frac{2}{\sqrt{5}}}$

Substitute in (2): $\boxed{\therefore \frac{d\theta}{dt}=-\frac{1}{2}(4)^{\frac{-3}{2}}(4)\left(\frac{2}{\sqrt{5}}\right)^2=\frac{-1}{5}}$

Behavior of functions

LESSON 1

Revision on previous year

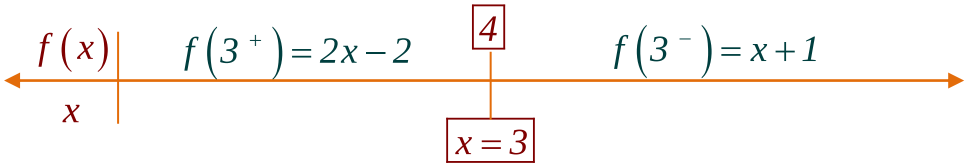
Continuity over point

*Last year, we have taken the Continuity of a function over a point which States that a function **F** is said to be continuous at a point $x = a$, if the Following condition exist :* $f(a) = \lim_{x \to a} f(a^-) = \lim_{x \to a} f(a^+)$

Example (1)

Let $f(x) = \begin{cases} x+1 & x \geq 3 \\ 2x-2 & x < 3 \end{cases}$, then discuss the continuity at $x = 3$

Answer



$\because f(3) = 3 + 1 = 4$

For $\lim_{x \to 3} f(x)$ Left limit : $f(3^-) = \lim_{x \to 3^-} 2x - 2 = 2(3) - 2 = 4$

Right limit : $f(3^+) = \lim_{x \to 3^+} x + 1 = 3 + 1 = 4$

$\therefore \; f(3^-) = f(3^+) = f(3) \Rightarrow \therefore f(3) = \lim_{x \to 3} f(x) = 4 \Rightarrow \therefore f(x)$ is continous at $x = 3$

Continuity over interval

(1) Any Constant function is continuous over R
(2) Any Polynomial function is continuous over R
(3) Any Sin or Cos function is continuous over R
(4) Any Exponential function is continuous over R
(5) Any Logarithmic function is continuous over R^+
(6) Any Fractional function is continuous over $R - \{denominator = 0\}$

Differentiability

A function is said to be differentiable at $x = a$ if $f'(a^-) = f'(a^+)$

$$\text{Where } f'(a^-) = f'(a^+) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

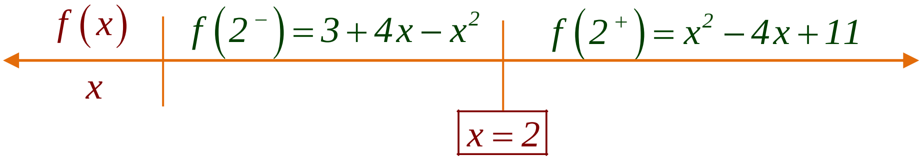
Remark If $f(x)$ is **differentiable** at $x = a$, then it is **continuous** at $x = a$.

Example (2)

Discuss the differentiability of $f(x)=\begin{cases} 3+4x-x^2 & x<2 \\ x^2-4x+11 & x\ge 2 \end{cases}$ at $x=2$

Answer

$$\because \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$



For $f'(2^+): \lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} \Rightarrow \lim_{h\to 0^+} \frac{(2+h)^2-4(2+h)+11-\left[(2)^2-4(2)+11\right]}{h}$

$\therefore \lim_{h\to 0^+} \frac{4+4h+h^2-8-4h+11-4+8-11}{h} \Rightarrow \lim_{h\to 0^+} \frac{h^2}{h} \Rightarrow \lim_{h\to 0^+} h=0$

For $f'(2^-): \lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} \Rightarrow \lim_{h\to 0^-} \frac{3+4(2+h)-(2+h)^2-\left[3+4(2)-(2)^2\right]}{h}$

$\therefore \lim_{h\to 0^-} \frac{3+8+4h-4-4h-h^2-9}{h} \Rightarrow \lim_{h\to 0^-} \frac{-h^2}{h} \Rightarrow \lim_{h\to 0^-} -h=0$

So $\because f'(2^+)=f'(2^-) \Rightarrow$ then $f(x)$ is differentiable at $x=2$ and its derivative is $f'(2)=0$

Behavior of functions

has four cases

Case (1) : Discuss the increasing and decreasing function

Case (2) : Discuss the local maximum and minimum function

Case (3) : Discuss the Absolute maximum and minimum

Case (4) : Discuss the Concavity and Convexity of the function

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Case (1) : Discuss the increasing and decreasing function

(1) If $f'(x) \geq 0$ in $]a,b[$, then $f(x)$ is an increasing function on this interval

(2) If $f'(x) < 0$ in $]a,b[$, then $f(x)$ is a decreasing function on this interval

Note $f'(x)$ *here means the slope of the function*

Steps

(1) Find the first derivative of the given function $f'(x)$.

(2) Find the Critical points using $f'(x) = 0$ which means to get the values of x .

(3) Find the interval over which the derivative is more than Zero .... Then for this interval function is increasing .

(4) Find the interval over which the derivative is less than Zero .... Then for this interval function is decreasing .

Example (1)

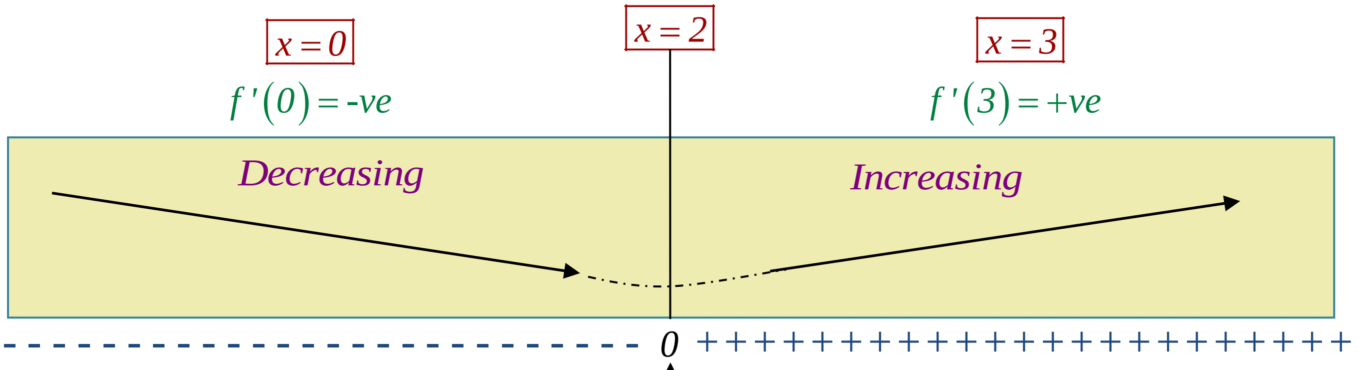
Determine the increasing and decreasing intervals for the function : $f(x) = x^2 - 4x + 5$

Answer

$\because$ *$f(x)$ is differentiable and continous on R*

(1) $f'(x) = 2x - 4$ *(2) for* $f'(x) = 0 \Rightarrow 2x - 4 = 0 \Rightarrow \therefore 2x = 4$

Then $x = 2$ *is the only critical point* $\Rightarrow$ *(3) take a point before and after* $x = 2$



$f(x)$ is decreasing over on $]-\infty, 2[$

$f(x)$ is decreasing over on $]-\infty, 2[$

$f(x)$ has a local minimum at $x = 2$

Very important note

Critical point occurs in one of the following cases

(1) **For polynomial functions**

(i) $f(x)$ is continous and $f'(x)=0$

(ii) The sign of curvature changes from increasing to decreasing or vice versa

(2) **For fractional functions** (derivatives)

(i) $f(x)$ is continous and $f'(x)=0$

(ii) $f'(x)$ doesn't exist except the set of zeroes of the denominator of the original fn

(3) **For double functions**

(i) $f(x)$ must be continous and not differentiable $\left[f'(a^-)\neq f'(a^+)\right]$

(ii) $f'(x)=0$ or $f'(x)$ doesn't exist

To understand what we are doing