Example (15)

If $xy = \sin x \cos x$, then prove that: $x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy = 0$

Answer

$xy = \frac{1}{2}\sin 2x$, so differentiate with respect to x : $x\frac{dy}{dx} + y = \cos 2x$

Differentiate again with respect to x : $x\frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = -2\sin 2x$

$\therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = -4\sin x\cos x \Rightarrow x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4\sin x\cos x = 0 \Rightarrow \therefore x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 4xy = 0$

Example (16)

If $y = \sec x$, then prove that : $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y^2\left(3y^2 - 2\right)$

Answer

Differentiate with respect to x : $\boxed{\therefore \frac{dy}{dx} = \sec x\tan x - - - (1)}$

Notes

$\because 1 + \tan^2 x = \sec^2 x$

$\therefore \tan^2 x - \sec^2 x = -1$

$\therefore \frac{d^2y}{dx^2} = \sec x\sec^2 x + \tan x\left[\sec x\tan x\right]$

$\therefore \frac{d^2y}{dx^2} = \sec^3 x + \sec x\tan^2 x = \sec x\left[\sec^2 x + \tan^2 x\right] = \sec x\left[\sec^2 x + \sec^2 x - 1\right]$

$\boxed{\therefore \frac{d^2y}{dx^2} = y\left[2y^2 - 1\right] - - - (2)}$, then from (1) and (2) and by solving

$\therefore L.H.S : y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = y^2\left[2y^2 - 1\right] + \left[\sec x\tan x\right]^2 = y^2\left[2y^2 - 1\right] + \sec^2 x\tan^2 x$

$\therefore y^2\left[2y^2 - 1\right] + \sec^2 x\left(\sec^2 x - 1\right) = y^2\left[2y^2 - 1\right] + y^2\left(y^2 - 1\right) = y^2\left[\left(2y^2 - 1 + y^2 - 1\right)\right]$

$\therefore L.H.S \Rightarrow y^2\left[3y^2 - 2\right] = R.H.S.$

Example (17)

If $y = a\,Cos\,x^2 + b\,Sin\,x^2$, then prove that : $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4x^3y = 0$

Answer

Differentiate with respect to x : $\dfrac{dy}{dx} = -a\,Sin\,x^2\,[2x] + b\,Cos\,x^2\,[2x]$

$$\therefore \frac{dy}{dx} = 2bx\,Cos\,x^2 - 2ax\,Sin\,x^2 = 2x\left[b\,Cos\,x^2 - a\,Sin\,x^2\right] --- (1)$$

$$\therefore \frac{d^2y}{dx^2} = 2x\left[-b\,Sin\,x^2\,(2x) - a\,Cos\,x^2\,(2x)\right] + 2\left[b\,Cos\,x^2 - a\,Sin\,x^2\right]$$

$$\therefore \frac{d^2y}{dx^2} = 2x\left[-2xb\,Sin\,x^2 - 2xa\,Cos\,x^2\right] + 2\left[b\,Cos\,x^2 - a\,Sin\,x^2\right]$$

$$\therefore \frac{d^2y}{dx^2} = -4x^2\left[b\,Sin\,x^2 + a\,Cos\,x^2\right] + 2\left[b\,Cos\,x^2 - a\,Sin\,x^2\right] --- (2)$$

Multiply both sides by x :

$$\therefore x\frac{d^2y}{dx^2} = -4x^3\left[b\,Sin\,x^2 + a\,Cos\,x^2\right] + 2x\left[b\,Cos\,x^2 - a\,Sin\,x^2\right]$$

$$\therefore x\frac{d^2y}{dx^2} = -4x^3y + \frac{dy}{dx} \Rightarrow \therefore x\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4x^3y = 0$$

Example (18)

If $y = Cos\,2x$, then prove that : $\left(\dfrac{d^2y}{dx^2}\right)^2 + 4\left(\dfrac{dy}{dx}\right)^2 = 16$

Answer

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Example (19)

Find the rate of change of $\sqrt{8+x^2}$ with respect to $\frac{x}{x+1}$ at $x=1$

Answer

Assume $y=\sqrt{8+x^2}=\left(8+x^2\right)^{\frac{1}{2}}$ and $Z=\frac{x}{x+1}$

$$\frac{dy}{dx}=\frac{1}{2}\left(8+x^2\right)^{\frac{-1}{2}}(2x)=\frac{x}{\left(8+x^2\right)^{\frac{1}{2}}} \quad \text{and} \quad \frac{dz}{dx}=\frac{(x+1)-x}{(x+1)^2}=\frac{1}{(x+1)^2}$$

$$\frac{dy}{dz}=\frac{dy}{dx}\times\frac{dx}{dz}=\frac{x}{\left(8+x^2\right)^{\frac{1}{2}}}\times(x+1)^2=\frac{x(x+1)^2}{\sqrt{\left(8+x^2\right)}}$$

So at $x=1$: $\left(\frac{dy}{dz}\right)_{x=1}=\frac{(1)(1+1)^2}{\sqrt{(8+1)}}=\frac{4}{3}$

Example (20)

If $xy=a$ where a is a real positive number and $\frac{d^2y}{dx^2}\times\frac{d^2x}{dy^2}>\frac{dy}{dx}\times\frac{dx}{dy}$

Then find the interval for which $\boldsymbol{a}$ belongs

Answer

$$y=ax^{-1}\Rightarrow \frac{dy}{dx}=-ax^{-2} \Rightarrow \frac{d^2y}{dx^2}=2ax^{-3}$$

$$x=ay^{-1}\Rightarrow \frac{dx}{dy}=-ay^{-2} \Rightarrow \frac{d^2x}{dy^2}=2ay^{-3}$$

$$\therefore 2ax^{-3}\times 2ay^{-3}>-ax^{-2}\times -ay^{-2} \Rightarrow 4a^2(xy)^{-3}>a^2(xy)^{-2}$$

$$\therefore 4>\frac{(xy)^{-2}}{(xy)^{-3}} \Rightarrow xy<4 \Rightarrow \therefore a<4 \Rightarrow \therefore a\in\left]0,4\right[$$

Example (21)

If $y=Cos^2x$, prove that : $\frac{d^2y}{dx^2}+4y-2=0$

Answer

Differentiate with respect to x : $\frac{dy}{dx}=2Cos\,x(-Sin\,x)=-\boxed{2\,Sin\,x\,Cos\,x}=-Sin\,2x$

$$L.H.S.:-2Cos\,2x+4Cos^2x-2=-2Cos\,2x+2\left(2Cos^2x-1\right)=-2Cos\,2x+2Cos\,2x=0$$

Example (22)

If $y = Z^2 + 3Z + 7$, $Z = Sin\,3x$, then prove that : $\frac{d^2y}{d\,x^2}$ at $x = \frac{\pi}{3}$

Answer

$$\frac{dy}{dz} = 2Z + 3 \quad \& \quad \frac{dz}{dx} = 3Cos3x \;\Rightarrow \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = (2Z + 3)(3Cos3x)$$

$$\therefore \frac{dy}{dx} = (2Sin3x + 3)(3Cos3x) = 6\,Sin3xCos3x + Cos3x = 3Sin6x + Cos3x$$

Differentiate with respect to x : $\frac{d^2y}{dx^2} = 18Cos6x - 3Sin3x$

At $x = \frac{\pi}{3}$: $\therefore \frac{d^2y}{dx^2} = 18$

Example (23)

If $y = Sin^3x$, $Z = Cos^3x$, then prove that : $\frac{d^2y}{dz^2} = \frac{1}{3Sin\,x\,Cos^4x}$

Answer

LESSON 2

Geometric applications on derivatives

Introduction

We know that the equation of the straight line : $y = mx + c$

Where $m = \text{the slope} = \dfrac{y_2 - y_1}{x_2 - x_1}$ **and** C : the y – intercept **are given**

And , the equation of straight line if one point and a slope are given

So, if $m = \text{given}$ & $\text{point} = (x_1, y_1) \Rightarrow$ $y - y_1 = m(x - x_1)$

And , the equation of straight line if two points (x_1, y_1) & (x_2, x_1) are given :

$$\therefore \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

Notes

(1) $m = \tan\theta \Rightarrow$ if θ is the +ve direction of x – axis is given .

(2) If two lines are parallel $(L_1 \parallel L_2) \Rightarrow$ $\therefore m_1 = m_2$

(3) If two lines are perpendicular $(L_1 \perp L_2) \Rightarrow$ $\therefore m_1 = \dfrac{-1}{m_2} \Rightarrow$ " Normal slopes"

(4) If the line is parallel to x – axis ($\perp$ to y – axis) $\Rightarrow$ $\therefore m = 0$

(5) If the line is parallel to y – axis ($\perp$ to x – axis) $\Rightarrow$ $\therefore m = \dfrac{1}{0}$

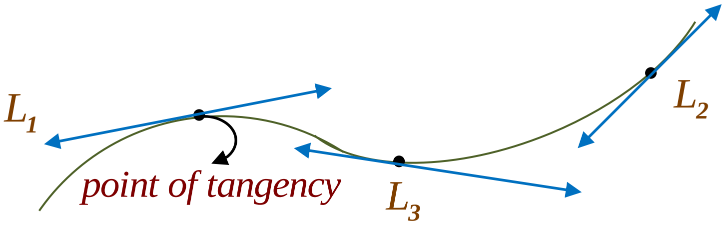
(6) If the line intersects x - axis $\Rightarrow$ means $y = 0 \rightarrow (a, 0)$

(7) If the line intersects y - axis $\Rightarrow$ means $x = 0 \rightarrow (0, b)$

(8) If a line intersects x , y axis at the same time, then the equation of line is $\dfrac{x}{a} + \dfrac{y}{b} = 1$

But what if

We want to find the equation of line passes through a point on a curve :



Then the equation of line here is called: **The equation of tangent line on the curve**

Its Rule

$$y - y_1 = m(x - x_1)$$

Where:

$$m = Tan\theta = \frac{dy}{dx} = f'(x) = \text{Slope of tangent line} \quad \text{and} \quad (x_1, y_1) \text{ is called the point of tangency}$$

Remarks

(1) The equation of the normal line to the tangent is : $y - y_1 = -\frac{1}{m}(x - x_1)$

(2) $$m = \left(\frac{dy}{dx}\right)_{(x,y)} = \begin{cases} 0 & \text{If the tangent is } // \text{ to } x-\text{axis} \\ Tan\ \theta & \text{If } \theta \text{ makes a } +ve \text{ angle with the } x-\text{axis} \\ \frac{1}{0} & \text{If the tangent is } \perp \text{ to } x-\text{axis} \end{cases}$$

Examples

Example (1)

Find the points on the curve $y = x^2 - 4x + 3$, at which the tangent to this curve is :

(1) Parallel to the $x-axis$

(2) Parallel to the line: $y = 2x + 3$

(3) Perpendicular to the line: $6y = 3x + 5$

Answer

(1) Parallel to $x-axis$ means that $\frac{dy}{dx} = 0 \Rightarrow \therefore \frac{dy}{dx} = 2x - 4 \Rightarrow 2x - 4 = 0 \Rightarrow \boxed{\therefore\ x = 2}$

Subsitute in the orignal : $y = (2)^2 - 4(2) + 3 = -1$

Then the point that the tangent passes with is $(2,-1)$

(2) $m = \frac{2}{1}$ is the slope of line $y = 2x + 3 \Rightarrow \therefore 2x = 6 \Rightarrow \boxed{\therefore\ x = 3}$

Substitute in the original equation: $y = (3)^2 - 4(3) + 3 = 0 \Rightarrow$ thus the point is $(3,0)$

(3) $m = \frac{3}{6} = \frac{1}{2}$ is the slope of $6y = 3x + 5 \Rightarrow$ then $\boxed{m_\perp = -2}$

$\therefore \frac{dy}{dx} = 2x - 4 = -2 \Rightarrow 2x = 2 \Rightarrow \boxed{\therefore\ x = 1}$

Subtitute in the equation : $y = (1)^2 - 4(1) + 3 = 0 \Rightarrow$ thus the point is $(1,0)$

Example (2)

Find the slope of the tangent to the curve of the function $x^2 + 3xy + 5y^2 = 3$ at $x = 2$ which lies on it.

Answer

At $x = 2 \Rightarrow \therefore (2)^2 + 3(2)y + 5y^2 = 3 \Rightarrow \therefore 5y^2 + 6y + 4 - 3 = 0 \Rightarrow \therefore 5y^2 + 6y + 1 = 0$

$\therefore (5y + 1)(y + 1) = 0 \Rightarrow \boxed{\therefore\ y = -\frac{1}{5}}$, $\boxed{y = -1}$

$\because$ The two points $\left(2, -\frac{1}{5}\right)$ & $(2,-1)$ are the points of tangency

<u>To find the slope:</u> differentiate w.r. to $x \Rightarrow \therefore 2x + 3x\frac{dy}{dx} + 3y + 10y\frac{dy}{dx} = 0$

$\therefore 3x\frac{dy}{dx} + 10y\frac{dy}{dx} = -(2x + 3y) \Rightarrow \therefore \frac{dy}{dx}(3x + 10y) = -(2x + 3y) \Rightarrow \boxed{\therefore \frac{dy}{dx} = \frac{-(2x + 3y)}{(3x + 10y)}}$

So, at $\left(2, -\frac{1}{5}\right)$: the slope $= \left(\frac{dy}{dx}\right)_{\left(2,-\frac{1}{5}\right)} = \frac{-\left(2(2) + 3\left(-\frac{1}{5}\right)\right)}{3(2) + 10\left(\frac{-1}{5}\right)} = -0.85$

At $(2,-1)$: the slope $\left(\frac{dy}{dx}\right)_{(2,-1)} = \frac{-(2(2) + 3(-1))}{3(2) + 10(-1)} = \frac{-1}{-4} = \frac{1}{4}$

Example (3)

Find the point on the curve $y = e^x$ such that its tangent line is parallel to the line $y = 2x$

Answer

$\because$ The tangent line of the curve is parallel to the line $y = 2x \Rightarrow \therefore$ they have the same slope

The slope of the curve $\dfrac{dy}{dx} = e^x$ and the slope of the line $\dfrac{dy}{dx} = 2x$

$\therefore\ e^x = 2\ \ (\ln \text{ both sides}) \Rightarrow \therefore\ \ln e^x = \ln 2 \Rightarrow \therefore\ x \ln e = \ln 2 \Rightarrow \boxed{\therefore\ x = \ln 2}$

Then substitute in the curve : $\boxed{\therefore\ y = e^{\ln 2} = 2}$ then the point is $(\ln 2, \ln 2)$

Example (4)

Find the equation of tangent to the curve $y = \log(3x+1)^4$ at $x = 1$

Answer

At $x = 1 \Rightarrow \therefore\ y = \log 4^4 = 4\log 4 \Rightarrow$ then $(1, 4\log 4)$ is the point of tangency

To get the slope of the tangent : let $y = 4\log(3x+1)$

$$\therefore \frac{dy}{dx} = \frac{4\times 3}{(3x+1)\ln 10} = \frac{12}{(3x+1)\ln 10} \Rightarrow \text{ at } x = 1 \Rightarrow \boxed{\therefore \frac{dy}{dx} = \frac{3}{\ln 10}}$$

Then the equation of tangent line : $y - y_1 = m(x - x_1) \Rightarrow \therefore\ y - 4\log 4 = \dfrac{3}{\ln 10}(x-1)$

Note : in order to change $\ln x \xrightarrow{To} \log x$

$\boxed{\text{Rule}}$ $\ln 10 = \dfrac{1}{\log_{10} e} = \dfrac{1}{\log e}$

$\therefore\ y - 4\log 4 = 3\log e\,(x-1) \Rightarrow \therefore\ y - 4\log 4 = 3x\log e - 3\log e$

$$\boxed{\therefore\ y - 3x\log e - 4\log 4 + 3\log e = 0}$$

Example (5)

Find the equation of the tangent and the normal to the curve $y = e^x \operatorname{Cos} x$ at the point $(0,1)$

Answer

$\because (0,1) \in$ curve and $m = \dfrac{dy}{dx} = e^x[-\operatorname{Sin} x] + \operatorname{Cos} x\,[e^x] = -e^x \operatorname{Sin} x + e^x \operatorname{Cos} x$

Then the slope of the tangent at $(0,1)$: $\left(\dfrac{dy}{dx}\right)_{x=0} = 1$

$\therefore$ The equation of the tangent line: $y - y_1 = m(x - x_1)$

$\therefore\ y - 1 = x \rightarrow \boxed{\therefore\ y - x - 1 = 0}$

And $\because$ the slope $= 1$, then the slope of the normal is -1

$\therefore$ The equation of the normal line : $y - 1 = -x \rightarrow \boxed{\therefore\ y + x - 1 = 0}$

Example (6)

Find the equation of the tangent to the curve of the function $xy+e^y=e$ at $x=0$ which lies on it. Then find y'' at the point of tangency . .

Answer

At $x=0 \Rightarrow \therefore e^y=e \Rightarrow \therefore \boxed{y=1} \Rightarrow$ then $(0,1)$ is the point of tangency

Then differentiate with respect to x: $\therefore xy'+y+y'e^y=0 \Rightarrow \therefore xy'+y'e^y=-y$

$$\therefore y'\left[x+e^y\right]=-y \Rightarrow \boxed{\therefore y'=\frac{-y}{x+e^y}---(1)}$$

Then at $(0,1)$ the slope of the tangent $\boxed{y'=\frac{-1}{e}}$

$\because$ The equation of the tangent line: $y-y_1=m(x-x_1)$

$$\therefore y-1=\frac{-1}{e}x \ (\text{multiply by } e) \Rightarrow \boxed{\therefore ey+x-e=0}$$

To get y'' from (1): $\therefore y''=\dfrac{(x+e^y)(-y')+y\left(1+y'e^y\right)}{(x+e^y)^2}$

$$\text{At } (0,1): \therefore y''=\frac{(e)\left(\frac{1}{e}\right)+\left(1+\left(\frac{-1}{e}\right)(e)\right)}{(e)^2} \Rightarrow \boxed{\therefore y''=\frac{1}{e^2}}$$

Example (7)

Find in terms of π, the equation of the tangent to the curve $y=Cos(x+y)$ whose tangent slope equals $\frac{-1}{2}$ such that $0\le x\le\pi$

Answer

$\because \frac{dy}{dx}=Tan\theta=\frac{-1}{2}$ "given" , so we have to find the point of tangency

And $\because y=Cos(x+y)---(1) \Rightarrow \therefore \frac{dy}{dx}=-\left[Sin(x+y)\right]\left(1+\frac{dy}{dx}\right)$

When $\frac{dy}{dx}=\frac{-1}{2} \Rightarrow \therefore \frac{-1}{2}=-\left[Sin(x+y)\right]\left(1-\frac{1}{2}\right) \Rightarrow \therefore \frac{1}{2}=\left[Sin(x+y)\right]\left(\frac{1}{2}\right) \Rightarrow \therefore Sin(x+y)=1$

$\therefore x+y=Sin^{-1}1=90^o$ "agreed" $---(2) \Rightarrow$ substitute in (1): $y=Cos\,90^o=0$

Then from (2): $x+0=90^o \Rightarrow \boxed{\therefore x=90^o}$, then the point of tangency is $(90^o,0)$

$$\therefore \text{Equation of the tangent is}: y-0=-\frac{1}{2}\left(x-\frac{\pi}{2}\right) \Rightarrow y=\frac{-1}{2}x+\frac{\pi}{4} \Rightarrow \boxed{\therefore 4y+2x-\pi=0}$$

Example (8)

Find the equation of the tangent to the curve $y=\frac{Tan\,x}{1-Tan^2x}$ at point $x=\frac{\pi}{3}$

Answer

$$\because Tan\,2x=\frac{2Tan\,x}{1-Tan^2x} \Rightarrow \therefore \frac{1}{2}Tan\,2x=\frac{Tan\,x}{1-Tan^2x} \Rightarrow \therefore y=\frac{1}{2}Tan\,2x$$

The equation of the tangent line : $y-y_1=m(x-x_1)$ where

$$x_1=\frac{\pi}{3} \quad \& \quad y_1=\frac{1}{2}Tan\,2\left(\frac{\pi}{3}\right)=-\frac{\sqrt{3}}{2} \quad and \quad m=\frac{dy}{dx}=Sec^2 2x$$

$$So\ at\quad x=\frac{\pi}{3} \Rightarrow Slope=\left(\frac{dy}{dx}\right)_{x=\frac{\pi}{3}}=Sec^2 2\left(\frac{\pi}{3}\right)=4$$

$$\therefore\ y+\frac{\sqrt{3}}{2}=4\left(x-\frac{\pi}{3}\right) \Rightarrow y=4x-\frac{4\pi}{3}-\frac{\sqrt{3}}{2} \quad (\times 6) \Rightarrow \boxed{\therefore\ 6y=24x-8\pi-3\sqrt{3}}$$

Example (9)

Find the equation of tangent and normal to the curve whose points are : $x=t^2+2t+3$ and $y=2t^3-6t+1$ at $t=0$ which lie on it

Answer

To get the point of tangency

At $t=0 \Rightarrow \therefore\ x=3$ and $y=1 \Rightarrow \therefore\ (3,1)$ is the point of tangency

To get the slope of the tangent $\boxed{\frac{dy}{dx}}$

$$\because \frac{dx}{dt}=2t+2=2(t+1) \quad and \quad \because \frac{dy}{dt}=6t^2-6=6(t^2-1)$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}=\frac{6(t^2-1)}{2(t+1)}=\frac{3(t+1)(t-1)}{(t+1)}=3(t-1)$$

Then at $t=0$ the slope of the tangent $\boxed{\frac{dy}{dx}=-3}$

$\because$ The equation of the tangent line: $y-y_1=m(x-x_1) \Rightarrow \therefore\ y-1=-3(x-3)$

$$\boxed{\therefore\ y+3x-10=0}$$

To get the normal $\because\ y'=\frac{1}{3} \Rightarrow \therefore$ The equation of the normal line:

$$\therefore\ y-1=\frac{1}{3}(x-3) \quad (multiply\ by\ 3) \Rightarrow \boxed{\therefore\ 3y-x=0}$$

Example (10)

Find the equation of tangent and normal to the curve whose points are $x = Sec^2\theta - 1$, $y = Tan\theta$ at $\theta = \frac{-\pi}{4}$ which lie on it

Answer

To get the point of tangency

At $\theta = \frac{-\pi}{4} \Rightarrow \therefore\ x = Sec^2\frac{-\pi}{4} - 1 = 1$ and $y = Tan\frac{-\pi}{4} = -1 \Rightarrow \therefore (1,-1)$ is the point of tangency

To get the slope of the tangent $\boxed{\frac{dy}{dx}}$

$\because \frac{dx}{d\theta} = 2(Sec\theta)Sec\theta Tan\theta = 2Sec^2\theta Tan\theta$ and $\because \frac{dy}{d\theta} = Sec^2\theta$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{Sec^2\theta}{2Sec^2\theta Tan\theta} = \frac{1}{2}Cot\theta$$

Then at $\theta = \frac{-\pi}{4}$ the slope of the tangent $\boxed{\frac{dy}{dx} = \frac{1}{2}Cot(-45^o) = \frac{-1}{2}}$

$\because$ The equation of the tangent line: $y + 1 = \frac{-1}{2}(x - 1)$ (multiply by 2) $\Rightarrow \boxed{\therefore\ 2y + x + 1 = 0}$

To get the normal $\because\ y' = 2 \Rightarrow \therefore$ the equation of the normal line:

$\therefore\ y + 1 = 2(x - 1) \Rightarrow \boxed{\therefore\ y - 2x + 3 = 0}$

Example (11)

If the curve $y = 3x^2 - 7x + 4$ intersects the x - axis in two points A and B , then prove that the two tangents at A and B are orthogonal .

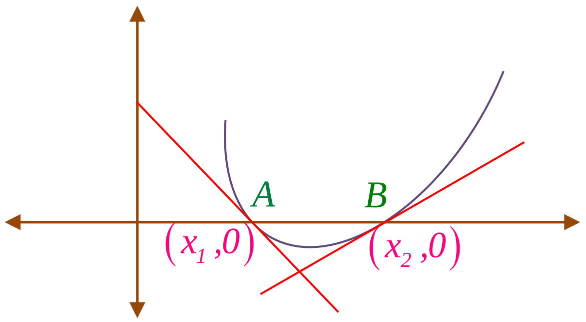
Answer

$\because$ The curve intersects the x - axis at A , B , then

Let $(x, 0)$ be the point of tangency between the tangent and the curve , then at : $(x_1, 0)$:

$$3x_1^2 - 7x_1 + 4 = 0 \Rightarrow (3x_1 - 4)(x_1 - 1) = 0$$

Then $\boxed{x_1 = \frac{4}{3}}$ or $\boxed{x_1 = 1}$



Then we have two tangents , their point of tangency are $\left(\frac{4}{3}, 0\right)$ and $(1, 0)$

So for the curve : $y = 3x^2 - 7x + 4 \Rightarrow$ the slope of the tangent is : $\frac{dy}{dx} = 6x - 7$

At $\left(\frac{4}{3}, 0\right)$: $\frac{dy}{dx} = 6\left(\frac{4}{3}\right) - 7 = 1$ and at $(1, 0)$: $\frac{dy}{dx} = 6(1) - 7 = -1$

So $\because$ the product of both slope of the two tangents is -1, then they are orthogonal

Example (12)

Prove that the two curves $y=2x^2-3x+8$ and $y=x^2-3x+9$ intersect orthogonally at x=1

Answer

To prove that the two curves intersect orthogonally :

First *: we have to prove that both curve intersect at a common point.*

Second *: prove that the product of their slopes = -1*

For $y=2x^2-3x+8$:

Substitute by $x=1$ in the equation : $y=2(1)^2-3(1)+8=7$

So the curve passes through $(1,7)$

Slope of the tangent : $\frac{dy}{dx}=4x-3$, so at $x=1$: $\left(\frac{dy}{dx}\right)_{x=1}=4(1)-3=1$

For $y=x^2-3x+9$:

by substituting in the equation by $x=1$: $y=(1)^2-3(1)+9=7$

So the curve passes through $(1,7)$

The slope of tangent : $\left(\frac{dy}{dx}\right)=2x-3$, so at $x=1$: $\left(\frac{dy}{dx}\right)_{x=1}=2(1)-3=-1$

$\therefore$ The two curves intersect at $(1,7)$ and their tangents are perpendicular

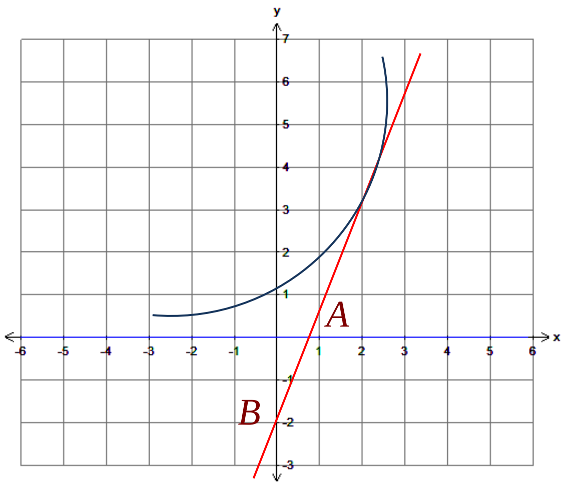
Then the two curves intersect orthogonally.

Example (13)

If the tangent to the curve $y=e^x$ at point $(2,e^2)$ cuts the x - axis at A and y - axis at B , then find the length $\overline{AB}$.

Answer

$(2,\ e^2)\in$ curve , then it is a point of tangency



To get the equation of tangent

The slope of the tangent at: $\frac{dy}{dx}=e^x$

At point $(2,\ e^2)$: $\left(\frac{dy}{dx}\right)_{(2,\ e^2)}=e^2$

Then the equation of the tangent is :

$y-e^2=e^2(x-2)\Rightarrow \boxed{y-e^2x+e^2=0}$

And $\because$ The tangent cuts the x - axis at A and y - axis at B:

Put $y=0 \Rightarrow \therefore\ e^2x=e^2 \Rightarrow \boxed{\therefore\ x=1} \Rightarrow \boxed{\therefore\ A=(1,0)}$

Put $x=0 \Rightarrow \therefore\ y=-e^2 \Rightarrow \boxed{\therefore\ B=(0,-e^2)}$

Then the length of $\overline{AB}=\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}=\sqrt{(-e^2)^2+(-1)^2}=\sqrt{e^4+1}\simeq 7.546$ unit length

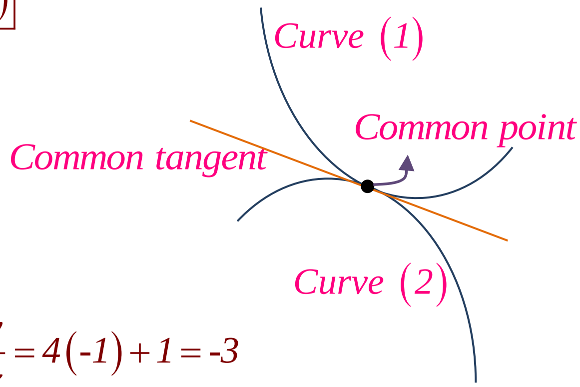
Example (14)

Prove that : the two curves $y = 2x^2 + x + 1$ and $y = x^2 - x$ **are tangential (touch each other)**, and find the equation of the common tangent at the point of tangency.

Answer

Tangential (touch each other) means that they have a **common point and tangent**

$\boxed{y = 2x^2 + x + 1 ---(1)}$ & $\boxed{y = x^2 - x ---(2)}$



Substitute (1) and (2) : $2x^2 + x + 1 = x^2 - x$

$x^2 + 2x + 1 = 0 \to (x+1)(x+1) = 0 \Rightarrow \therefore x = -1$

Substitute to get y : $\therefore y = (-1)^2 - (-1) = 2$

$\therefore$ Both curves touch each other in one point $(-1,2)$

For $y = 2x^2 + x + 1$: $\dfrac{dy}{dx} = 4x + 1$ at $(-1,2) \Rightarrow \dfrac{dy}{dx} = 4(-1) + 1 = -3$

For $y = x^2 - x$: $\dfrac{dy}{dx} = 2x - 1$ at $(-1,2) \Rightarrow \dfrac{dy}{dx} = 2(-1) - 1 = -3$

So , both curves touch at one point $(-1,2)$ and the slope of their tangents is the same $= -3$

Then they have a common tangent :

Its equation is : $y - y_1 = m(x - x_1) \Rightarrow y - 2 = -3(x+1) \Rightarrow y = -3x - 3 + 2 \Rightarrow \boxed{\therefore y = -3x - 1}$

Example (15)

Prove that : the curve $\left(\dfrac{x}{a}\right)^n + \left(\dfrac{y}{b}\right)^n = 2$ touches the line $\dfrac{x}{a} + \dfrac{y}{b} = 2$ at (a,b) from any value of n

Answer

To prove that the curve and the line touch each other, we must prove that they have the same point in common and the same tangent .

So substitute by point (a,b) in the curve : $\therefore \left(\dfrac{a}{a}\right)^n + \left(\dfrac{b}{b}\right)^n = 1^n + 1^n = 2 = R.H.S$

This means that (a,b) is the point of tangency between the curve and the line

Slope of the curve

let the curve be $\dfrac{1}{a^n} x^n + \dfrac{1}{b^n} y^n = 2$

Differentiate w.r. to x: $\therefore \dfrac{n}{a^n} x^{n-1} + \dfrac{n}{b^n} y^{n-1} \dfrac{dy}{dx} = 0$

$\therefore \dfrac{n}{b^n} y^{n-1} \dfrac{dy}{dx} = \dfrac{-n}{a^n} x^{n-1} \Rightarrow \therefore a^n y^{n-1} \dfrac{dy}{dx} = -b^n x^{n-1}$

$\boxed{\therefore \dfrac{dy}{dx} = \dfrac{-b^n x^{n-1}}{a^n y^{n-1}}}$

At (a,b): $\therefore \dfrac{dy}{dx} = \dfrac{-b^n a^{n-1}}{a^n b^{n-1}} = \dfrac{-b}{a} \Rightarrow \boxed{\therefore \dfrac{dy}{dx} = \dfrac{-b}{a}}$

Slope of the line

let the line be $\dfrac{1}{a} x + \dfrac{1}{b} y = 2$

Differentiate w.r. to x: $\therefore \dfrac{1}{a} + \dfrac{1}{b} \dfrac{dy}{dx} = 0$

$\therefore \dfrac{1}{b} \dfrac{dy}{dx} = \dfrac{-1}{a} \Rightarrow \boxed{\therefore \dfrac{dy}{dx} = \dfrac{-b}{a}}$

So, $\because$ both curve and line touch each other at (a,b) for any values of n

Example (16)

Find the equation of the tangent and the normal to the curve $y = x^2 + 2x - 3$ at its points of intersection with the $y - x + 1 = 0$

Answer

To find the point of intersection between the curve and the line .

$$\boxed{y = x^2 + 2x - 3 - - - (1)} \quad \& \quad \boxed{y = x - 1 - - - (2)}$$

Substitute (2) in (1): $\therefore\ x - 1 = x^2 + 2x - 3 \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0$

$\therefore$ When $\boxed{x = -2 \Rightarrow y = -3}$ and when $\boxed{x = 1 \Rightarrow y = 0}$

$\therefore$ The points of intersection between the curve and the line are : $(-2,-3)$ & $(1,0)$

Then the equation of tangent line of the curve at $(-2,-3)$: $\because \frac{dy}{dx} = 2x + 2$

At $(-2,-3) \Rightarrow \boxed{\therefore \frac{dy}{dx} = 2(-2) + 2 = -2} \Rightarrow \therefore\ y + 3 = -2(x+2) \Rightarrow \boxed{\therefore\ y = -2x - 7}$

$\therefore$ The equation of normal line with $m = \frac{1}{2}$ is $y + 3 = \frac{1}{2}(x+2) \Rightarrow \boxed{\therefore\ 2y = x - 4}$

The equation of the tangent line of the curve at $(1,0)$: $\because \frac{dy}{dx} = 2x + 2 \Rightarrow \boxed{\therefore\ y = 4x - 4}$

$\therefore$ The equation of normal line : $y = -\frac{1}{4}(x-1)\ (\times 4) \Rightarrow 4y = -x + 1 \Rightarrow \boxed{\therefore\ 4y + x - 1 = 0}$

Example (17)

If $x \in [0, \pi]$, then find a point on a curve $y = \sin 2x - \cos x$ such that the tangent at it to the curve makes an angle of measure 135° with the +ve direction of x – Axis , find also the equation of this tangent.

Answer

$\because$ The slope of the tangent $= \tan\theta \Rightarrow \boxed{\therefore \frac{dy}{dx} = \tan 135^\circ = -1 - - - (1)}$

And $\boxed{\because \frac{dy}{dx} = 2\cos 2x + \sin x - - - (2)}$, then from $(1)\&(2)$: $\therefore\ 2\cos 2x + \sin x = -1$

$\because \cos 2x = 1 - 2\sin^2 x \Rightarrow \therefore\ 2(1 - 2\sin^2 x) + \sin x = -1 \Rightarrow \therefore\ 2 - 4\sin^2 x + \sin x + 1 = 0$

$\therefore\ -4\sin^2 x + \sin x + 3 = 0\ (\times -1) \Rightarrow \therefore 4\sin^2 x - \sin x - 3 = 0 \Rightarrow (4\sin x + 3)(\sin x - 1) = 0$

$\sin x = -\frac{3}{4} \Rightarrow x = \sin^{-1} -\frac{3}{4} = -48^\circ 35'$ "refused" or $\sin x = 1$

Or $x = 180^\circ - (-48^\circ 35') = 228^\circ 35'$ " refused" | $\boxed{x = \sin^{-1} 1 = \frac{\pi}{2}}$

So $x = \frac{\pi}{2}$ "agreed" $\rightarrow$ substitute to get $y \Rightarrow \boxed{\therefore\ y = \sin\pi - \cos\frac{\pi}{2} = 0}$

Then the point is $\left(\frac{\pi}{2}, 0\right)$: so the equation of tangent line : $y = -1\left(x - \frac{\pi}{2}\right) \Rightarrow \boxed{\therefore\ y = -x + \frac{\pi}{2}}$

Example (18)

Find the points on the curve $y=\dfrac{(x+2)^2}{x+3}$, $x\neq -3$ such that the normal to the curve at this point is parallel to the straight line : $4x+3y-5=0$

Answer

$\because$ The slope of the normal to the curve and the line are parallel , then both have the same slope :

The slope of the line : $4x+3y-5=0 \Rightarrow 3y=5-4x$ is $\dfrac{-4}{3}$

Then the slope of the normal is the same as the line $\dfrac{-4}{3}$

Then the slope of the tangent to the curve is $\dfrac{3}{4}$

The slope of the curve : $\dfrac{dy}{dx}=\dfrac{2(x+3)(x+2)-(x+2)^2}{(x+3)^2}=\dfrac{(x+2)\left[2(x+3)-(x+2)\right]}{(x+3)^2}$

Then $\dfrac{(x+2)\left[2x+6-x-2\right]}{(x+3)^2}=\dfrac{(x+2)(x+4)}{(x+3)^2}=\dfrac{x^2+6x+8}{x^2+6x+9}$

$\therefore \dfrac{x^2+6x+8}{x^2+6x+9}=\dfrac{3}{4} \Rightarrow 4x^2+24x+32=3x^2+18x+27$

$\therefore x^2+6x+5=0 \Rightarrow (x+5)(x+1)=0 \Rightarrow \boxed{x=-5}$ or $\boxed{x=-1}$

By substitute in the curve : when $x=-5 \Rightarrow \boxed{y=\dfrac{-9}{2}}$ and when $x=-1 \Rightarrow \boxed{y=\dfrac{1}{2}}$

Then the points on the curve are $\left(-1,\dfrac{1}{2}\right)$ and $\left(-5,\dfrac{-9}{2}\right)$

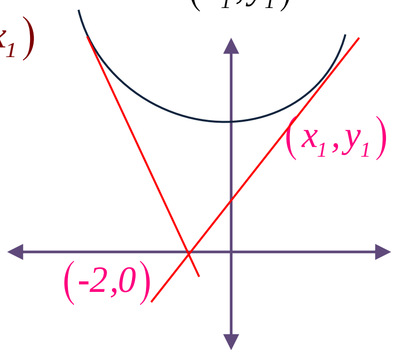
Example (19)

Find the two equations of the two tangents to the curve $y^2 = 8x$ which passes through $(-2,0)$ and prove that the two tangents are perpendicular .

Answer

$(-2,0) \notin$ curve , so let the point of tangency between the tangent and the curve be (x_1, y_1)

So the equation of the first tangent with the curve : $y - y_1 = m(x - x_1)$



So to get y_1 at x_1 : $\boxed{y_1^{\,2} = 8x_1 \;---(1)}$

and the slope of the tangent on the curve : $2y\dfrac{dy}{dx} = 8 \xrightarrow{\;\to\;} \dfrac{dy}{dx} = \dfrac{4}{y}$

At $(x_1, y_1) \to \dfrac{dy}{dx} = \dfrac{4}{y_1}$

But the slope of the tangent line passing through $(-2,0)$ & (x_1, y_1) is : $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1}{x_1 + 2}$

So , $\because$ both slopes are equal , then : $\dfrac{y_1}{x_1 + 2} = \dfrac{4}{y_1} \Rightarrow \boxed{\therefore\; y_1^{\,2} = 4x_1 + 8 \;---(2)}$

Substitue (2) in (1): $8x_1 = 4x_1 + 8 \xrightarrow{\;\to\;} 4x_1 = 8 \Rightarrow \boxed{\therefore\; x_1 = 2}$

$\therefore$ Substitute in (2) : $y_1^{\,2} = 16 \xrightarrow{\;\to\;} \boxed{\therefore\; y_1 = \pm 4}$

Then the two points of tangency are : $(2,4)$ and $(2,-4)$

At point$(2,4)$: $\because \dfrac{dy}{dx} = \dfrac{4}{y} = \dfrac{4}{4} = 1$

$\therefore$ Equation of the first tangent : $y - 4 = 1(x - 2) \Rightarrow y = x - 2 + 4 \Rightarrow \boxed{\therefore\; y = x + 2}$

At point $(2,-4)$: $\because \dfrac{dy}{dx} = \dfrac{4}{y} = \dfrac{4}{-4} = -1$

$\therefore$ Equation of the first tangent : $y + 4 = -1(x - 2) \Rightarrow y = -x + 2 - 4 \Rightarrow \boxed{\therefore\; y = -x - 2}$

Example (20)

Find the equation of the tangent to the circle $x^2 - y^2 = 16$ *which passes through the point* $(2,-2)$

Answer

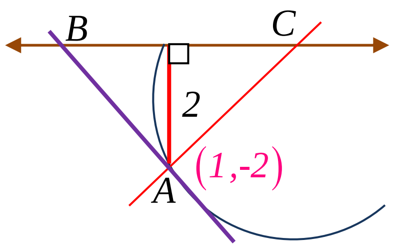
$Ans:\ 3y - 5x + 16 = 0$

Example (21)

If the tangent to the curve : $y^2+2y-4x+4=0$ at point $A(1,-2)$ intersects the x - axis at B and the normal to this curve at A intersects the x - axis at C , find :

(i) The length of $\overline{BC}$ (ii) The surface area of Δ ABC

Answer



$(1,-2)\in$ curve , so we have to find the tangent to the curve.

To get the equation of tangent at A :

the slope of the tangent at A : $2y\frac{dy}{dx}+2\frac{dy}{dx}-4=0 \quad (\div 2)$

Then $y\frac{dy}{dx}+\frac{dy}{dx}=2 \Rightarrow \frac{dy}{dx}(y+1)=2 \Rightarrow \frac{dy}{dx}=\frac{2}{y+1}$

At $A(1,-2)$: $\left(\frac{dy}{dx}\right)_{(1,-2)}=-2$

Then the equation of the tangent passing through A is : $y+2=-2(x-1)\Rightarrow \boxed{y+2x=0}$

And $\because$ point B is on the x - axis and lies on the tangent $\Rightarrow$ put $\boxed{y=0} \Rightarrow \boxed{\therefore x=0}$

Then $B=(0,0)$

To get the Normal on A :

the slope : $\frac{dy}{dx}=\frac{1}{2}$ and its equations is : $y+2=\frac{1}{2}(x-1)\Rightarrow \boxed{2y-x+5=0}$

And $\because$ point C is on the x - axis and lies on the Normal $\Rightarrow$ put $y=0 \Rightarrow \therefore x=5$

Then $C=(5,0)$

Then the distance between $\overline{BC}$ is 5 units Or we can get it from the rule $\sqrt{(y_2-y_1)^2+(x_2-x_1)^2}$

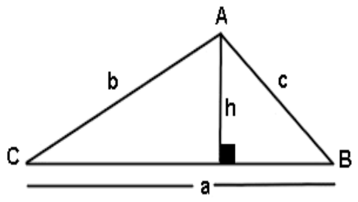
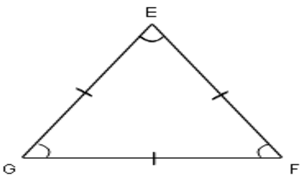
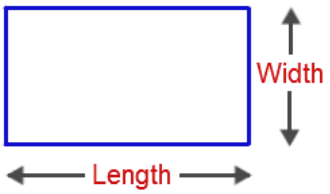
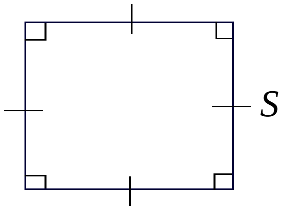
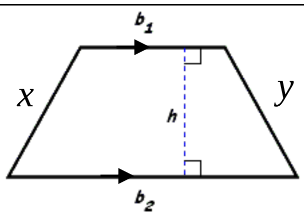
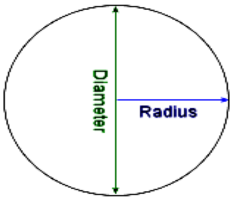
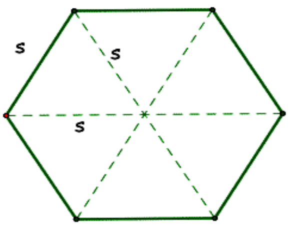
Then the area of Δ ABC $=\frac{1}{2}\times 5\times 2=5$ square units

3 LESSON

Related time rates

Remember that

Two dimensions shapes

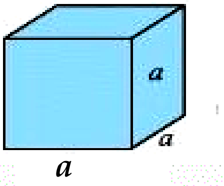
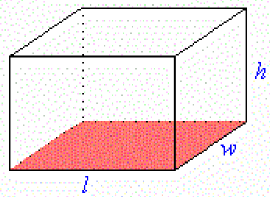
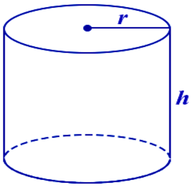
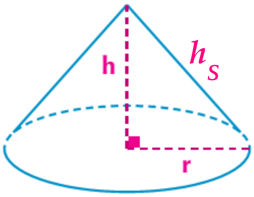
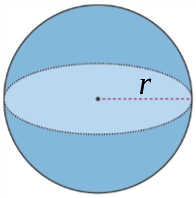
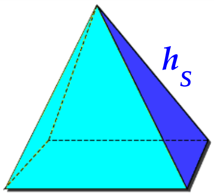
Name	Figure	Area	Perimeter
Scalene Triangle		$\frac{1}{2} \times base \times height$ $\frac{1}{2} ab \, Sin C$	$a+b+c$
Equilateral Triangle		$\frac{\sqrt{3}}{4}(side)^2$	$3 \times Side$
Rectangle		$length \times width$	$2\left[length + width\right]$
Square		$(Side)^2$	$4 \times Side$
Trapezuim Trapeziod		$\frac{1}{2}(b_1 + b_2) \times h$	$(x + y + b_1 + b_2)$
Circle		πR^2	$2\pi R$
Hexagon		$\frac{3\sqrt{3}}{2}(S)^2$	$6 \times Side$

Three dimensions shapes

Basic RULE

$Volume = Area\ of\ the\ base \times Height$

$Total\ surface\ area = (Perimeter\ of\ the\ base \times height) + area\ of\ the\ 2\ parallel\ bases$

Name	Figure	Area	Volume
Cube		$6(side)^2$	$(Side)^3$
Cuboid Parallelepiped		$2(L+w)h+2Lw$	$L \times W \times H$
Cylinder		$2\pi Rh + 2\pi R^2$	$\pi R^2 h$
Cone		$\pi R h_s + \pi R^2$	$\frac{1}{3}\pi R^2 h$
Sphere		$4\pi R^2$	$\frac{4}{3}\pi R^3$
Pyramid		$\frac{1}{2}[\text{base perimeter} \times h_s]$ $+ \text{area of base}$	$\frac{1}{3}[\text{area of base} \times h]$

What do we mean by time rates ?

When a circular lamina is heated, then its radius increases and also its surface area . This increasement , either in its radius or in its surface area , changes with respect to the time , for which the lamina is heated. Let the radius of the lamina be R and its surface area is S , then $\frac{dr}{dt}$ and $\frac{ds}{dt}$ (T - is the time) are called the related time rates of R and S .