$(11)\ y = \dfrac{e^{-x}\,Cos^2 x}{x^2+x+1}$

Answer

$\boxed{Note}$ *The function is complicated product and quotient , so ln both sides to simplify*

$$\therefore\ ln\,y = ln\,\frac{e^{-x}\,Cos^2 x}{x^2+x+1} \ \Rightarrow\ \therefore\ ln\,y = ln\,e^{-x}\,Cos^2 x - ln\left(x^2+x+1\right)$$

$$\therefore\ ln\,y = ln\,e^{-x} + ln\,(Cos\,x)^2 - ln\left(x^2+x+1\right) \Rightarrow\ \therefore\ ln\,y = -x\,ln\,e + 2\,ln\,(Cos\,x) - ln\left(x^2+x+1\right)$$

$$\therefore\ ln\,y = -x + 2\,ln\,(Cos\,x) - ln\left(x^2+x+1\right)$$

Then differentiate w.r.to x : $\therefore\ \dfrac{y'}{y} = -1 + 2\left[\dfrac{-Sin\,x}{Cos\,x}\right] - \dfrac{2x+1}{x^2+x+1} = -1 - 2Tan\,x - \dfrac{2x+1}{x^2+x+1}$

$$\therefore\ y' = -y\left[1 + 2Tan\,x + \frac{2x+1}{x^2+x+1}\right] \ \Rightarrow\ \boxed{\therefore\ y' = \frac{-e^{-x}\,Cos^2 x}{x^2+x+1}\left[1 + 2Tan\,x + \frac{2x+1}{x^2+x+1}\right]}$$

Lesson 6: Exponential and logarithmic function as a limit

Introduction last year, you have taken $\lim_{x \to a} f(x)$ and we have learnt that its concept is used to describe the behavior of a function as its input either "gets close" to some point, or as the argument becomes arbitrarily large

Also, we knew that if the value was unspecified $\left[\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, \infty \times 0\right]$, we used to solve this problem either by factorization , conjugate , long division …… etc.

Rules

$(1)\ \lim_{x \to 0} \frac{\operatorname{Sin} a x}{b x} = \frac{a}{b}$ $\quad (2)\ \lim_{x \to 0} \frac{\operatorname{Tan} a x}{b x} = \frac{a}{b}$ $\quad (3)\ \lim_{x \to \infty} \frac{1}{x} = 0$

Also, we have taken the rule of compound interest which has the form $f(t) = m_o\left(1 + \frac{rate}{t}\right)^t$

Now we are going to discuss the relation between the natural exponential function $(e = 2.718\ldots)$ and the limit by the following life example . Suppose that you have one pound in a bank which offers you 100 % interest rate. If you want to know how much your saving is considering that :

Annually with rate 100 %	Semi Annually with rate 100 %	Quarter Annually with rate 100 %
$(1+1)^1 = 2$	$\left(1+\frac{1}{2}\right)^2 = 2.25$	$\left(1+\frac{1}{4}\right)^4 = 2.44\ldots$
Monthly with rate 100 %	**daily with rate 100 %**	**hourly with rate 100 %**
$\left(1+\frac{1}{12}\right)^{12} = 2.613\ldots \simeq e$	$\left(1+\frac{1}{365}\right)^{365} = 2.714\ldots \simeq e$	$\left(1+\frac{1}{365 \times 24}\right)^{365 \times 24} = 2.718\ldots \simeq e$

Minute with rate 100 %	Secondly with rate 100 %
$\left(1+\frac{1}{365 \times 24 \times 60}\right)^{365 \times 24 \times 60} = 2.7182\ldots \simeq e$	$\left(1+\frac{1}{365 \times 24 \times 60 \times 60}\right)^{365 \times 24 \times 60 \times 60} = 2.7182\ldots \simeq e$

The relation between e and limit

From what we have introduced before, we can say that the relation between the natural exponent and the limit can be identified by:

Rule

$$\underbrace{e=\lim_{x\to\infty}}_{must}\left(1+\frac{1}{\underbrace{x}_{must}}\right)^{\overset{must}{x}} \quad \textbf{or} \quad e=\lim_{x\to 0}(1+x)^{\frac{1}{x}}$$

Note : $\lim\limits_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}\neq 1$, this is because it doesn't represent a number, but it represents idea

Examples

Find the value of each of the following limits :

Example (1)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}$$

Answer

$\boxed{\text{Steps}}$ (1) Substitute $f(\infty)=1^{\infty}\ \Rightarrow$ so we can use the law of e

(2) Be sure that all variable are the same . $\Rightarrow \lim\limits_{\boxed{x}\to\infty}\left[\left(1+\frac{1}{\boxed{x}}\right)^{\boxed{x}}\right]^{3}$

(3) Put the limit in the form of $e^{something}$. $\Rightarrow\ e^{3}$

$$\boxed{\text{Then}\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3x}=e^{3}}$$

Example (2)

$$\lim_{x\to 0}(1+x)^{\frac{2}{x}}$$

Answer

$$f(0)=1^{\infty}\ \Rightarrow\ \therefore\ \lim_{x\to 0}(1+x)^{\frac{2}{x}}=\lim_{x\to 0}\left[(1+x)^{\frac{1}{x}}\right]^{2}=e^{2}$$

Example (3)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+3}$$

Answer

$$\because\ f(\infty)=1^{\infty}\ \Rightarrow\ \therefore\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x+3}=\lim_{\boxed{x}\to\infty}\left(1+\frac{1}{\boxed{x}}\right)^{\boxed{x}}\times\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{3}=e\times(1)^{3}=e$$

Example (4)

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x+5}$$

Answer

$$\because\ f(\infty)=1^{\infty}\ \Rightarrow\ \therefore\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x+5}=\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x}\times\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5}$$

$$\therefore\ \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{2x+5}=\left[\lim_{\boxed{x}\to\infty}\left(1+\frac{1}{\boxed{x}}\right)^{\boxed{x}}\right]^{2}\times\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{5}=e^{2}\times(1)^{5}=e^{2}$$

Example (5)

$$\lim_{x\to\infty}\left(1-\frac{5}{x}\right)^{x}$$

Answer

$$\because\ f(\infty)=1^{\infty}\ \Rightarrow\ \therefore\ \lim_{x\to\infty}\left(1-\frac{5}{x}\right)^{x}=\lim_{x\to\infty}\left(1+\frac{-5}{x}\right)^{x}$$

Let $\frac{-5}{x}=y\ \Rightarrow$ so when $x\to\infty$ then $y\to 0$

$$\therefore\ \lim_{y\to 0}(1+y)^{\frac{-5}{y}}\ \Rightarrow\ \therefore\ \lim_{y\to 0}\left[(1+y)^{\frac{1}{y}}\right]^{-5}=e^{-5}=\frac{1}{e^{5}}$$

Draft

$$\frac{-5}{x}=\frac{y}{1}\ \Rightarrow\ \therefore\ x=\frac{-5}{y}$$

Example (6)

$$\lim_{x\to 0}(1+4x)^{\frac{3}{7x}}$$

Answer

$$f(0)=1^{\infty}\ \Rightarrow\ \therefore\ \lim_{x\to 0}(1+4x)^{\frac{3}{7x}}=\lim_{\boxed{4x}\to 0}\left[(1+\boxed{4x})^{\frac{1}{\boxed{4x}}}\right]^{\frac{3\times 4}{7}}=e^{\frac{12}{7}}$$

Example (7)

$$\lim_{x \to 0}\left(\frac{1+2x}{1-3x}\right)^{\frac{1}{x}}$$

Answer

$$\because\ f(0)=1^{\infty} \Rightarrow \therefore \lim_{x \to 0}\left(\frac{1+2x}{1-3x}\right)^{\frac{1}{x}} = \lim_{x \to 0}\left[1+2x\right]^{\frac{1}{x}} \div \lim_{x \to 0}\left[1-3x\right]^{\frac{1}{x}}$$

$$\lim_{x \to 0}\left(\frac{1+2x}{1-3x}\right)^{\frac{1}{x}} = \lim_{\boxed{2x} \to 0}\left[\left(1+\boxed{2x}\right)^{\frac{1}{\boxed{2x}}}\right]^{2} \div \lim_{\boxed{-3x} \to 0}\left[\left(1+\boxed{-3x}\right)^{\frac{1}{\boxed{-3x}}}\right]^{-3} = \frac{e^{2}}{e^{-3}} = e^{5}$$

Example (8)

$$\lim_{x \to \infty}\left(\frac{x+2}{x-1}\right)^{x+4}$$

Answer

$\because\ f(\infty) \neq 1^{\infty}$, so we have to fix the shape

$$\therefore \lim_{x \to \infty}\left(\frac{x+2}{x-1}\right)^{x+4} = \lim_{x \to \infty}\left(\frac{x-1+3}{x-1}\right)^{x+4} = \lim_{x \to \infty}\left(\frac{x-1}{x-1}+\frac{3}{x-1}\right)^{x+4} = \lim_{x \to \infty}\left(1+\frac{3}{x-1}\right)^{x+4}$$

Let $\dfrac{3}{x-1} = y \Rightarrow$ so when $x \to \infty$ then $y \to 0$

$\boxed{x}$

Draft

$$\frac{3}{x-1} = \frac{y}{1} \Rightarrow \therefore x-1 = \frac{3}{y}$$

$$\therefore x = \frac{3}{y}+1$$

$$\therefore \lim_{y \to 0}(1+y)^{\frac{3}{y}+1+4} \Rightarrow \therefore \lim_{y \to 0}(1+y)^{\frac{3}{y}+5}$$

$$\therefore \lim_{y \to 0}(1+y)^{\frac{3}{y}} \times \lim_{y \to 0}(1+y)^{5}$$

$$\therefore \lim_{y \to 0}\left[(1+y)^{\frac{1}{y}}\right]^{3} \times \lim_{y \to 0}(1+y)^{5} = e^{3} \times 1 = e^{3}$$

Example (9)

$$\lim_{x \to 0}(1+x)^{-1-\frac{1}{x}}$$

Answer

....

....

....

....

....

.... $Ans : \dfrac{1}{e}$

Example (10)

$$\lim_{x \to \infty} \left(\frac{x}{1+x} \right)^x$$

Answer

$\because\ f(\infty) \neq 1^{\infty}$, so we have to fix the shape

$$\therefore \lim_{x \to \infty} \left(\frac{x}{1+x} \right)^x = \lim_{x \to \infty} \left(\frac{1+x-1}{1+x} \right)^x = \lim_{x \to \infty} \left(\frac{1+x}{1+x} + \frac{-1}{1+x} \right)^x = \lim_{x \to \infty} \left(1 + \frac{-1}{1+x} \right)^x$$

Let $\frac{-1}{1+x} = y \Rightarrow$ so when $x \to \infty$ then $y \to 0$

$\boxed{x}$

$$\therefore \lim_{y \to 0} (1+y)^{\frac{-1}{y}-1} \Rightarrow \therefore \lim_{y \to 0} (1+y)^{\frac{-1}{y}} \times \lim_{y \to 0} (1+y)^{-1}$$

$$\therefore \lim_{y \to 0} \left[(1+y)^{\frac{1}{y}} \right]^{-1} \times \lim_{y \to 0} (1+y)^{-1} = e^{-1} \times 1 = e^{-1} = \frac{1}{e}$$

Draft

$$\frac{-1}{1+x} = \frac{y}{1} \Rightarrow \therefore x+1 = \frac{-1}{y}$$

$$\therefore x = \frac{-1}{y} - 1$$

Example (11)

$$\lim_{x \to \infty} \left(\frac{2x+5}{2x+1} \right)^{x+2}$$

Answer

$\because\ f(\infty) \neq 1^{\infty}$, so we have to fix the shape

$$\therefore \lim_{x \to \infty} \left(\frac{2x+5}{2x+1} \right)^{x+2} = \lim_{x \to \infty} \left(\frac{2x+1+4}{2x+1} \right)^{x+2} = \lim_{x \to \infty} \left(\frac{2x+1}{2x+1} + \frac{4}{2x+1} \right)^{x+2} = \lim_{x \to \infty} \left(1 + \frac{4}{2x+1} \right)^{x+2}$$

Let $\frac{4}{2x+1} = y \Rightarrow$ so when $x \to \infty$ then $y \to 0$

$\boxed{x}$

$$\therefore \lim_{y \to 0} (1+y)^{\frac{2}{y}-\frac{1}{2}+2} \Rightarrow \therefore \lim_{y \to 0} (1+y)^{\frac{2}{y}} \times \lim_{y \to 0} (1+y)^{\frac{3}{2}}$$

$$\therefore \lim_{y \to 0} \left[(1+y)^{\frac{1}{y}} \right]^{2} \times \lim_{y \to 0} (1+y)^{\frac{3}{2}} = e^2 \times 1 = e^2$$

Draft

$$\frac{4}{2x+1} = \frac{y}{1} \Rightarrow \therefore 2x+1 = \frac{4}{y}$$

$$\therefore 2x = \frac{4}{y} - 1 \Rightarrow x = \frac{2}{y} - \frac{1}{2}$$

Another relations with limit

The relation between a^x and limit

Rule

$$\boxed{\lim_{x \to 0} \frac{a^x - 1}{x} = \ln a}$$

must

must

must

The relation between log and limit

Rule

$$\boxed{\lim_{x \to 0} \frac{\log_a (x+1)}{x} = \log_a e}$$

must

must

must

$$\boxed{\lim_{x \to 0} \frac{\ln (x+1)}{x} = 1}$$

must

must

must

Very important remark :

If the limit was unspecified $\left(\frac{0}{0}, \frac{\pm \infty}{\pm \infty} \right)$ value after substitution, you can check your result by using l'HOPITAL's theorem which states that :

$$\boxed{\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{f'(x)}{g'(x)} \rightarrow \text{ then substitute by } x}$$

But you have to know that this check is for yourself only as l'HOPITAL's theorem is not required to use it at this year .

Examples

Example (1)

$$\lim_{x \to 0} \frac{a^{3x}-1}{x}$$

Answer

$$\because f(0)=\frac{0}{0}, \therefore \lim_{x \to 0} \frac{a^{3x}-1}{x} \Rightarrow \lim_{3x \to 0} \frac{a^{3x}-1}{3x} \times 3 = 3\ln a$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{a^{3x} \ln a \, [3]}{1}$$

$$\frac{f'(0)}{g'(0)} = 3\ln a$$

Example (2)

$$\lim_{x \to 0} \frac{e^{\operatorname{Sin} x}-1}{x}$$

Answer

$$\because f(0)=\frac{0}{0}, \therefore \lim_{\operatorname{Sin} x \to 0} \frac{e^{\operatorname{Sin} x}-1}{\operatorname{Sin} x} \times \frac{\operatorname{Sin} x}{x}$$

$$\therefore \lim_{\operatorname{Sin} x \to 0} \frac{e^{\operatorname{Sin} x}-1}{\operatorname{Sin} x} \times \lim_{x \to 0} \frac{\operatorname{Sin} x}{x} = \ln e \times 1 = 1$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{e^{\operatorname{Sin} x} \ln e \, [\operatorname{Cos} x]}{1}$$

$$\frac{f'(0)}{g'(0)} = e^{0} \, [\operatorname{Cos}(0)] = 1$$

Example (3)

$$\lim_{x \to 0} \frac{e^{x}-1}{\operatorname{Tan} x}$$

Answer

$$\because f(0)=\frac{0}{0}, \therefore \lim_{x \to 0} \frac{e^{x}-1}{\operatorname{Tan} x} \times \frac{x}{x}$$

$$\therefore \lim_{x \to 0} \frac{e^{x}-1}{x} \times \lim_{x \to 0} \frac{x}{\operatorname{Tan} x} = \ln e \times 1 = 1$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{e^{x}}{\operatorname{Sec}^2 x}$$

$$\frac{f'(0)}{g'(0)} = \frac{e^{0}}{\operatorname{Sec}^2(0)} = 1$$

Example (4)

$$\lim_{x \to 0} \frac{e^{4x} - 1}{\tan x}$$

Answer

$$\because \; f(0) = \frac{0}{0} \;,\; \therefore \; \lim_{x \to 0} \frac{e^{4x} - 1}{\tan x} \times \frac{4x}{4x}$$

$$\therefore \; \lim_{4x \to 0} \frac{e^{4x} - 1}{4x} \times \lim_{x \to 0} \frac{4x}{\tan x} = \ln e \times 4 = 4$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{4e^{4x}}{\sec^2 x}$$

$$\frac{f'(0)}{g'(0)} = \frac{4e^{0}}{\sec^2(0)} = 4$$

Example (5)

$$\lim_{x \to 0} \frac{3^{2+x} - 9}{x}$$

Answer

$$\because \; f(0) = \frac{0}{0} \;,\; \therefore \; \lim_{x \to 0} \frac{3^2 \times 3^x - 9}{x} \Rightarrow \lim_{x \to 0} \frac{9 \times 3^x - 9}{x}$$

$$\therefore \; 9 \lim_{x \to 0} \frac{3^x - 1}{x} = 9 \ln 3$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{3^{2+x} \ln 3}{1}$$

$$\frac{f'(0)}{g'(0)} = \frac{3^2 \ln 3}{1} = 9 \ln 3$$

Example (6)

$$\lim_{x \to 0} \frac{e^{x^2} - \cos x}{x^2}$$

Answer

$$\because \; f(0) = \frac{0}{0} \;,\; \therefore \; \lim_{x \to 0} \frac{e^{x^2} - 1 + 1 - \cos x}{x^2}$$

$$\therefore \; \lim_{x^2 \to 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x \to 0} \frac{1 - \cos x}{x^2}$$

$$\therefore \; \lim_{x^2 \to 0} \frac{e^{x^2} - 1}{x^2} + \lim_{x \to 0} \frac{1 - 1 + 2\sin^2 \frac{x}{2}}{x^2}$$

$$\therefore \; \lim_{x^2 \to 0} \frac{e^{x^2} - 1}{x^2} + 2 \lim_{x \to 0} \left[\frac{\sin \frac{x}{2}}{x} \right]^2 = \ln e + 2\left(\frac{1}{2}\right)^2 = \frac{3}{2}$$

Check for yourself

$$\frac{f'(x)}{g'(x)} = \frac{e^{x^2} \ln e [2x] + \sin x}{2x}$$

$$\frac{f'(x)}{g'(x)} = \frac{2x\, e^{x^2} + \sin x}{2x}$$

$$\frac{f'(0)}{g'(0)} = \frac{2x\, e^{x^2} + \sin x}{2x} = \frac{0}{0}$$

$$\frac{f''(x)}{g''(x)} = \frac{2x\, e^{x^2} \ln e [2x] + 2e^{x^2} + \cos x}{2}$$

$$\frac{f''(0)}{g''(0)} = \frac{2e^{0} + 1}{2} = \frac{3}{2}$$

Example (7)

$$\lim_{x\to 0}\frac{e^{Tan\,x}-e^{x}}{Tan\,x-x}$$

Answer

$$\because\ f(0)=\frac{0}{0}\ ,\ \therefore\ \lim_{x\to 0}\frac{e^{x}\left[e^{Tan\,x-x}-1\right]}{Tan\,x-x}$$

$$\therefore\ \lim_{x\to 0}e^{x}\times\lim_{Tan\,x-x\to 0}\frac{e^{Tan\,x-x}-1}{Tan\,x-x}=e^{0}\times\ln e=1$$

> *Check for yourself*
>
> In this problem, you will check by using l'hopital more than one time, which will be too long

---

Example (8)

$$\lim_{x\to 0}\frac{e^{x+3}-Sin\,x-e^{3}}{x}$$

Answer

$$\because\ f(0)=\frac{0}{0}\ ,\ \therefore\ \lim_{x\to 0}\frac{e^{x+3}-e^{3}}{x}-\frac{Sin\,x}{x}$$

$$\therefore\ \lim_{x\to 0}\frac{e^{3}\left[e^{x}-1\right]}{x}-\lim_{x\to 0}\frac{Sin\,x}{x}$$

$$\therefore\ e^{3}\lim_{x\to 0}\frac{e^{x}-1}{x}-\lim_{x\to 0}\frac{Sin\,x}{x}=e^{3}\ln e-1=e^{3}-1$$

> *Check for yourself*
>
> Don't ever write this in your Exam paper
>
> $$\frac{f'(x)}{g'(x)}=\frac{e^{x+3}\ln e-Cos\,x}{1}$$
>
> $$\frac{f'(0)}{g'(0)}=\frac{e^{3}-Cos(0)}{1}=e^{3}-1$$

Example (9)

$$\lim_{x\to 0}\frac{\ln(1+2x)}{x}$$

Answer

$$\because\ f(0)=\frac{0}{0}$$

$$\therefore\ \lim_{x\to 0}\frac{\ln(1+2x)}{x}=2\lim_{x\to 0}\frac{\ln(1+2x)}{2x}=2\times 1=2$$

> *Check for yourself*
>
> Don't ever write this in your Exam paper
>
> $$\frac{f'(x)}{g'(x)}=\frac{\left[\frac{2}{1+2x}\right]}{1}$$
>
> $$\frac{f'(0)}{g'(0)}=\frac{2}{1}=2$$

Example (10)

$$\lim_{x \to 0} \frac{\ln\left(1+3x^2\right)}{2x^2}$$

Answer

$$\because\ f(0)=\frac{0}{0}$$

$$\therefore\ \frac{1}{2}\lim_{x \to 0} \frac{\ln\left(1+3x^2\right)}{x^2} \Rightarrow \therefore\ \frac{3}{2}\lim_{3x^2 \to 0} \frac{\ln\left(1+3x^2\right)}{3x^2} = \frac{3}{2}$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{\left[\dfrac{6x}{1+3x^2}\right]}{4x} = \frac{6x}{4x\left(1+3x^2\right)}$$

$$\frac{f'(0)}{g'(0)} = \frac{0}{0} \Rightarrow \frac{f''(x)}{g''(x)} = \frac{6}{4+36x^2}$$

$$\frac{f''(0)}{g''(0)} = \frac{6}{4} = \frac{3}{2}$$

Example (11)

$$\lim_{x \to 0} \frac{\sqrt{1+x}-1}{\ln(1+x)}$$

Answer

$$\because\ f(0)=\frac{0}{0} \Rightarrow \therefore\ \lim_{x \to 0} \frac{\sqrt{1+x}-1}{\ln(1+x)} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$\therefore\ \lim_{x \to 0} \frac{x}{\left[\sqrt{1+x}+1\right]\ln(1+x)}$$

$$\therefore\ \lim_{x \to 0} \frac{x}{\ln(1+x)} \times \lim_{x \to 0} \frac{1}{\sqrt{1+x}+1} = 1 \times \frac{1}{2} = \frac{1}{2}$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{\left[\dfrac{1}{2\sqrt{1+x}}\right]}{\left[\dfrac{1}{1+x}\right]}$$

$$\frac{f'(0)}{g'(0)} = \frac{\frac{1}{2}}{1} = \frac{1}{2}$$

Example (12)

$$\lim_{x \to 0} \frac{\log 10 + \log\left(x+\dfrac{1}{10}\right)}{x}$$

Answer

$\boxed{\text{Note}}$ Same way of ln with another rule

$$\because\ f(0)=\frac{0}{0} \Rightarrow \therefore\ \lim_{x \to 0} \frac{\log 10\left(x+\dfrac{1}{10}\right)}{x}$$

$$\therefore\ \lim_{x \to 0} \frac{\log(10x+1)}{x}$$

$$\therefore 10 \lim_{10x \to 0} \frac{\log(10x+1)}{10x} = 10\log_{10} e \simeq 4.34$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{\left[\dfrac{1}{(x+0.1)\ln 10}\right]}{1}$$

$$\frac{f'(0)}{g'(0)} = \frac{1}{0.1\ln 10} \simeq 4.34$$

Example (13)

$$\lim_{x \to a} \frac{\ln x - \ln a}{x - a}$$

Answer

$\boxed{\text{Note}}$ The problem does not satisy the condition $x \to 0$

$$\because\ f(0) = \frac{0}{0} \Rightarrow \therefore \lim_{x-a \to 0} \frac{\ln x - \ln a}{x - a}$$

$$\therefore \text{let } x - a = y \Rightarrow \therefore \lim_{y \to 0} \frac{\ln(y+a) - \ln a}{y}$$

$$\therefore \lim_{y \to 0} \frac{\ln\left(\frac{y+a}{a}\right)}{y} \Rightarrow \therefore \lim_{y \to 0} \frac{\ln\left(\frac{y}{a}+1\right)}{y}$$

$$\therefore \lim_{\frac{y}{a} \to 0} \frac{\ln\left(\frac{y}{a}+1\right)}{\frac{y}{a} \times a} \Rightarrow \therefore \frac{1}{a} \lim_{\frac{y}{a} \to 0} \frac{\ln\left(\frac{y}{a}+1\right)}{\frac{y}{a}} = \frac{1}{a}$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{\left[\frac{1}{x}\right]}{1}$$

$$\frac{f'(a)}{g'(a)} = \frac{1}{a}$$

Example (14)

$$\lim_{x \to e} \frac{\ln x - 1}{x - e}$$

Answer

$\boxed{\text{Note}}$ The problem does not satisy the condition $x \to 0$

$$\because\ f(0) = \frac{0}{0} \Rightarrow \therefore \lim_{x-e \to 0} \frac{\ln x - 1}{x - e}$$

$$\therefore \text{let } x - e = y \Rightarrow \therefore \lim_{y \to 0} \frac{\ln(y+e) - 1}{y} = \frac{\ln(y+e) - \ln e}{y}$$

$$\therefore \lim_{y \to 0} \frac{\ln\left(\frac{y+e}{e}\right)}{y} \Rightarrow \therefore \lim_{y \to 0} \frac{\ln\left(\frac{y}{e}+1\right)}{y}$$

$$\therefore \lim_{\frac{y}{e} \to 0} \frac{\ln\left(\frac{y}{e}+1\right)}{\frac{y}{e} \times e} \Rightarrow \therefore \frac{1}{e} \lim_{\frac{y}{e} \to 0} \frac{\ln\left(\frac{y}{e}+1\right)}{\frac{y}{e}} = \frac{1}{e}$$

Check for yourself

Don't ever write this in your Exam paper

$$\frac{f'(x)}{g'(x)} = \frac{\left[\frac{1}{x}\right]}{1}$$

$$\frac{f'(e)}{g'(e)} = \frac{1}{e}$$

Summary of rules of differentiation

There are four cases for powers and bases

(1) $\boxed{\text{Constant}}$ **base**, $\boxed{\text{Constant}}$ **power** $\Rightarrow \dfrac{d}{dx}\left(a^{b}\right)=0$

(2) $\boxed{\text{Variable}}$ **base**, $\boxed{\text{Constant}}$ **power** $\Rightarrow \dfrac{d}{dx}\left[f(x)\right]^{b}=b\left[f(x)\right]^{b-1}\times f'(x)$

(3) $\boxed{\text{Constant}}$ **base**, $\boxed{\text{Variable}}$ **power** $\Rightarrow \dfrac{d}{dx}\,[a]^{f(x)}=[a]^{f(x)}\ln(a)\times f'(x)$

(4) $\boxed{\text{Variable}}$ **base**, $\boxed{\text{Variable}}$ **power** $\Rightarrow \dfrac{d}{dx}\left[f(x)\right]^{g(x)}=$ logarithmic differentiation (as it is complicated)

Rules

A function	Its derivative
(1) If $f(x)=Sec\,x$	$f'(x)=Tan\,x\,Sec\,x$
(2) If $f(x)=Cosec\,x$	$f'(x)=-\,Cosec\,x\,Cot\,x$
(3) If $f(x)=Cot\,x$	$f'(x)=-\,Cosec^{2}\,x$
(4) If $f(x)=Sec\,a\,x$	$f'(x)=a\,Tan\,a\,x\,Sec\,a\,x$
(5) If $f(x)=Cosec\,a\,x$	$f'(x)=-a\,Cosec\,a\,x\,Cot\,a\,x$
(6) If $f(x)=Cot\,a\,x$	$f'(x)=-a\,Cosec^{2}\,a\,x$
(7) If $f(x)=e^{x}$	$f'(x)=e^{x}$
(8) If $f(x)=e^{g(x)}$	$f'(x)=e^{g(x)}\left[g'(x)\right]$

Rules of limit

(1) $\lim\limits_{x\to 0}(1+x)^{\frac{1}{x}}=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}=e$

(2) $\lim\limits_{x\to 0}\dfrac{a^{x}-1}{x}=\ln a$

(3) $e=\lim\limits_{x\to\infty}\left(1+\dfrac{1}{x}\right)^{x}$

(4) $\lim\limits_{x\to 0}\dfrac{\log_{a}(x+1)}{x}=\log_{a}e$

Very important remark :

If the limit was unspecified $\left(\dfrac{0}{0},\dfrac{\pm\infty}{\pm\infty}\right)$ value after substitution, you can check your result by using l'HOPITAL's theorem which states that :

$\lim\limits_{x\to 0}\dfrac{f(x)}{g(x)}=\dfrac{f'(x)}{g'(x)}\rightarrow$ then substitute by x

But you have to know that this check is for yourself only as l'HOPITAL's theorem is not required to use it at this year .

The natural logarithmic function $\Rightarrow \ln x$

Rule – 1: If $\ln_e x = y \Rightarrow \boxed{\therefore e^y = x}$	Rule – 2: $\ln x = \ln_e x$, where $x \in R^+$
Rule – 3: $\ln e = 1$ (as they are inverse)	Rule – 4: $\ln x^{\boxed{n}} = \boxed{n} \ln x$
Rule – 5: $\log_a x = \dfrac{\ln x}{\ln a}$ (from log to ln)	Rule – 6: $\ln_e x = \dfrac{1}{\log_x e}$ (from ln to log)
Rule – 7: $\ln x\,y = \ln x + \ln y$	Rule – 8: $\ln \dfrac{x}{y} = \ln x - \ln y$
Rule – 9: when power is unknown (ln both sides)	Rule – 10: $\ln e^x = e^{\ln x} = x$

The derivative of $\boxed{a^x}$	The derivative of $\boxed{\ln x}$	The derivative of $\boxed{\log_a x}$
Rules (1) If $y = a^x$, $a \in R^+ - \{1\}$ Then $\boxed{\dfrac{dy}{dx} = a^x \ln a}$ (2) If $y = a^{bx}$, $a \in R^+ - \{1\}$ Then $\boxed{\dfrac{dy}{dx} = \left[a^{bx} \ln a\right] \times b}$ **Generally** If $y = a^{g(x)}$, $a \in R^+ - \{1\}$ $\boxed{\therefore \dfrac{dy}{dx} = \left[a^{g(x)} \ln a\right] \times g'(x)}$	**Rules** (1) If $y = \ln x$ or $y = \ln \lvert x \rvert$ Then $\boxed{\dfrac{dy}{dx} = \dfrac{1}{x}}$ (2) If $y = \ln ax$ Then $\boxed{\dfrac{dy}{dx} = \dfrac{a}{ax}}$ **Generally** If $y = \ln g(x)$ $\boxed{\therefore \dfrac{dy}{dx} = \dfrac{g'(x)}{g(x)} = \dfrac{\text{derivative}}{\text{itself}}}$	**Rules** (1) If $y = \log_a x$ Then $\boxed{\dfrac{dy}{dx} = \dfrac{1}{x \ln a}}$ (2) If $y = \log_a bx$ Then $\boxed{\dfrac{dy}{dx} = \dfrac{b}{bx \ln a}}$ **Generally** If $y = \log_a g(x)$ $\boxed{\dfrac{dy}{dx} = \dfrac{g'(x)}{g(x) \ln a} = \dfrac{\text{derivative}}{\text{itself} \times \ln \text{base}}}$

Complicated differentiation

Steps to solve complicated derivatives:

1. ***ln*** *the both sides of the equation and use the laws of logarithms to simplify.*
2. *Differentiate implicitly with respect to x .*
3. *Solve the resulting equation for y'.*

If $f(x) < 0$ for some values of x , then $\ln(x)$ is not defined, but we can write $\lvert y \rvert = \lvert f(x) \rvert$.

1 LESSON

Higher derivatives

Introduction

If $y=\frac{1}{4}x^4+2x^3-x^2+3x-1$

Then the first derivative is called: $y'=\frac{dy}{dx}=x^3+6x^2-2x+3$

The second derivative is called: $y''=\frac{d^2y}{dx^2}=3x^2+12x-2$

The third derivative is called: $y'''=\frac{d^3y}{dx^3}=6x+12$

The forth derivative is called: $y''''=\frac{d^4y}{dx^4}=6$

The fifth derivative is called: $y'''''=\frac{d^5y}{dx^5}=0$ and so on........

Chain rule

Also, we have taken before the chain rule,and we knew that there are two shapes of it .

First : Leibniz notation

If $y=f(z)$ and $z=g(x)$

$$\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$

Second : Prime notation

If $z(x)=f(g(x))$ or $z(x)=((f\circ g)(x))$

Then $z'(x)=f'(g(x))\times g'(x)$

Examples

Example (1)

If $y=Z^5$ and $Z=2x+3$, then find $\frac{d^2y}{dx^2}$

Answer

$$\frac{dy}{dz}=5Z^4 \text{ and } \frac{dz}{dx}=2 \Rightarrow \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=5Z^4\times 2=10Z^4$$

Then $\frac{dy}{dx}=10(2x+3)^4 \Rightarrow \frac{d^2y}{dx^2}=40(2x+3)^3(2)=80(2x+3)^3$

Example (2)

If $x = e^{2t}$ and $y = t^3$, then find $\frac{d^2y}{dx^2}$

Answer

$\boxed{\text{Note}}$ This equations are called parametric equations

$$\frac{dx}{dt} = 2e^{2t} \quad \text{and} \quad \frac{dy}{dt} = 3t^2 \Rightarrow \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{3t^2}{2e^{2t}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{2e^{2t}[6t] - 3t^2\left[4e^{2t}\right]}{\left[2e^{2t}\right]^2} = \frac{12te^{2t}\frac{dt}{dx} - 12t^2e^{2t}\frac{dt}{dx}}{4e^{4t}}$$

$$\therefore \frac{d^2y}{dx^2} = \frac{\frac{12te^{2t}}{2e^{2t}} - \frac{12t^2e^{2t}}{2e^{2t}}}{4e^{4t}} = \frac{6t - 6t^2}{4e^{4t}} = \frac{6t(1-t)}{4e^{4t}} = \frac{3t(1-t)}{2e^{4t}}$$

Example (3)

If $x = t^3$ and $y = 4\ln t$, then find $\frac{d^2y}{dx^2}$

Answer

$\boxed{\text{Note}}$ This equations are called parametric equations

$$\frac{dx}{dt} = 3t^2 \quad \text{and} \quad \frac{dy}{dt} = \frac{4}{t} \Rightarrow \therefore \frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = \frac{4}{t} \times \frac{1}{3t^2} = \frac{4}{3t^3} = \frac{4}{3}t^{-3}$$

$$\therefore \frac{d^2y}{dx^2} = -4t^{-4} \times \frac{dt}{dx} = \frac{-4}{t^4} \times \frac{1}{3t^2} = \frac{-4}{3t^6}$$

Example (4)

If $f(x) + x^2\left[f(x)\right]^3 = 10$ and $f(1) = 2$, then find $f'(1)$

Answer

Differentiate w. r. to x : $f'(x) + x^2\left[3\left(f(x)\right)^2 f'(x)\right] + \left[f(x)\right]^3[2x] = 0$

$$\therefore f'(x) + 3x^2\left(f(x)\right)^2 f'(x) + 2x\left[f(x)\right]^3 = 0$$

At $x = 1$: $\therefore f'(1) + 3(1)^2\left(f(1)\right)^2 f'(1) + 2(1)\left[f(1)\right]^3 = 0$

$$\therefore f'(1) + 3(2)^2 f'(1) + 2[2]^3 = 0 \Rightarrow \therefore f'(1) + 12f'(1) + 16 = 0$$

$$\therefore 13f'(1) = -16 \Rightarrow \boxed{\therefore f'(1) = \frac{-16}{13}}$$

Example (5)

If $f(x)=\dfrac{2}{x+1}$ and $g(x)=3x$, then find $\dfrac{d}{dx}\left[(f\circ g)(x)\right]$ at $x=-2$

Answer

$(f\circ g)(x)$ means $f(g(x))=\dfrac{2}{(3x+1)}=2(3x+1)^{-1}$

Differentiate w. r. to x : $\therefore\ \dfrac{d}{dx}\ f(g(x))=-2(3x+1)^{-2}[3]$

$$\therefore\ \frac{d}{dx}\ f(g(x))=\frac{-6}{(3x+1)^2}$$

$\therefore$ At $x=-2$: $\therefore\ \dfrac{d}{dx}\ f(g(-2))=\dfrac{-6}{(3(-2)+1)^2}=\dfrac{-6}{25}$

Example (6)

If $3y^4=4x^3$, then prove that : $y\,y''+3(y')^2=\dfrac{2x}{y^2}$

Answer

Differentiate with respect to x : $12y^3y'=12x^2 \Rightarrow \therefore\ y^3y'=x^2$

Differentiate again with respect to x : $y^3y''+y'(3y^2y')=2x \Rightarrow y^3y''+3y^2(y')^2=2x$

$\therefore\ y^2\left[y\,y''+3(y')^2\right]=2x \quad\Rightarrow\quad y\,y''+3(y')^2=\dfrac{2x}{y^2}$

Example (7)

If $x^2+y^2=25$, prove that $y''=\dfrac{-25}{y^3}$

Answer

Differentiate with respect to x

$2x+2y\,y'=0$ " $\div$ by 2" $\Rightarrow\ x+y\,y'=0 \Rightarrow \boxed{\therefore\ y'=\dfrac{-x}{y}---(1)}$

Differentiate again with respect to x : $\therefore$ $\boxed{y''=\dfrac{-y+x\,y'}{y^2}---(2)}$

So by substituting (1) in (2) :

$$y''=\frac{-y+x\left(\frac{-x}{y}\right)}{y^2}=\frac{-y-\frac{x^2}{y}}{y^2}=\frac{-\frac{y^2}{y}-\frac{x^2}{y}}{y^2}=\frac{-\frac{1}{y}\left(y^2+x^2\right)}{y^2}=\frac{-25}{y^3}$$

Example (8)

If $y = 8x^3 - 1$ and $Z = 6x^4$, then find $\dfrac{d^2z}{dy^2}$ at $x = 1.5$

Answer

$\dfrac{d^2z}{dy^2}$ means that we have to differentiate with respect to y

So for $\boxed{y = 8x^3 - 1} \Rightarrow \dfrac{dy}{dx} = 24x^2$ And for $\boxed{y = 6x^4} \Rightarrow \dfrac{dz}{dx} = 24x^3$

And $\because \dfrac{dz}{dy} = \dfrac{dz}{dx} \times \dfrac{dx}{dy} = \dfrac{24x^3}{24x^2} = x \Rightarrow$ then differentiate again with respect to y

$\therefore$ Then $\dfrac{d^2z}{dy^2} = \dfrac{dx}{dy} = \dfrac{1}{24x^2}$, so at $x = 1.5 \Rightarrow \dfrac{d^2z}{dy^2} = \dfrac{1}{24(1.5)^2} = \dfrac{1}{54}$

Example (9)

If $\dfrac{dz}{dx} = 3 + 2x$ and $\dfrac{dy}{dx} = x^2 - 3$, then find $\dfrac{d^2z}{dy^2}$ at $x = 2$

Answer

....

....

....

....

....

....

....

....

....

.... $\boxed{Ans: -26}$

Example (10)

If $y = (x^3 + 2)^7$, then prove that : $\dfrac{dx}{dy} = \dfrac{x^3 + 2}{21x^2 y}$

Answer

Differentiate with respect to x :

$$\frac{dy}{dx} = 7(x^3 + 2)^6 (3x^2) = 21x^2 (x^3 + 2)^6$$

$$\frac{dx}{dy} = \frac{1}{21x^2 (x^3 + 2)^6} = \frac{(x^3 + 2)}{21x^2 (x^3 + 2)^7} = \frac{x^3 + 2}{21x^2 y}$$

Example (11)

If $y = x\,Cos\,x + Sin\,x$, then prove that : $x\,y''' + x\,y' + 2y = 2\,Sin\,x$

Answer

$\because\ y' = -x\,Sin\,x + Cos\,x + Cos\,x = -x\,Sin\,x + 2Cos\,x$

And $y'' = -x\,Cos\,x - Sin\,x - 2Sin\,x = -\left(x\,Cos\,x + Sin\,x + 2Sin\,x\right)$

And $\because\ y = x\,Cos\,x + Sin\,x \ \Rightarrow\ y'' = -\left(y + 2Sin\,x\right)$

$\therefore\ y'' = -y - 2Sin\,x$ " Multiply both sides by x " $\Rightarrow\ x\,y'' = -x\,y - 2x\,Sin\,x$

So by differentiating again with respect to x :

$\therefore\ x\,y''' + y'' = -x\,y' - y - 2\left[x\,Cos\,x + Sin\,x\right] \Rightarrow\ x\,y''' + y'' = -x\,y' - y - 2y$

$\therefore\ x\,y''' - y - 2Sin\,x = -x\,y' - y - y \ \Rightarrow\ \therefore\ x\,y''' + x\,y' + 2y = 2\,Sin\,x$

Example (12)

If $y = \left(x^2 + 1\right)\left(x - 2\right)$ and $Z = \left(x + 7\right)\left(x - 2\right)$, then prove that : $\left(2x + 5\right)^3 \dfrac{d^2y}{dz^2} = 6x^2 + 30x - 22$

Answer

$$\frac{dy}{dx} = \left(x^2 + 1\right)\left(1\right) + \left(x - 2\right)\left(2x\right) = x^2 + 1 + 2x^2 - 4x = 3x^2 - 4x + 1$$

$$\frac{dz}{dx} = \left(x + 7\right)\left(1\right) + \left(x - 2\right)\left(1\right) = x + 7 + x - 2 = 2x + 5$$

$$\frac{dy}{dz} = \frac{dy}{dx} \times \frac{dx}{dz} = \frac{\left(3x^2 - 4x + 1\right)}{\left(2x + 5\right)}$$

So differentiate again with respect to Z :

$$\frac{d^2y}{dz^2} = \frac{\left(2x + 5\right)\left(6x - 4\right)\dfrac{dx}{dz} - \left(3x^2 - 4x + 1\right)\left(2\right)\dfrac{dx}{dz}}{\left(2x + 5\right)^2} = \frac{\dfrac{dx}{dz}\left[\left(2x + 5\right)\left(6x - 4\right) - 2\left(3x^2 - 4x + 1\right)\right]}{\left(2x + 5\right)^2}$$

$$\frac{d^2y}{dz^2} = \frac{12x^2 - 8x + 30x - 20 - 6x^2 + 8x - 2}{\left(2x + 5\right)\left(2x + 5\right)^2} = \frac{6x^2 + 30x - 22}{\left(2x + 5\right)\left(2x + 5\right)^2}$$

$$\therefore\ \left(2x + 5\right)^3 \frac{d^2y}{dz^2} = 6x^2 + 30x - 22$$

Example (13)

If $y = e^{-x}\sqrt{\frac{1+x}{1-x}}$ where $-1 < x < 1$, then prove that: $\left(1-x^2\right)y' = x^2\, y$

Answer

Note The function is complicated product and quotient , so ln both sides to simplify

$$\therefore\ \ln y = \ln e^{-x}\sqrt{\frac{1+x}{1-x}} \Rightarrow \therefore\ \ln y = \ln e^{-x} + \ln\left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} \Rightarrow \therefore\ \ln y = -x\ \ln e + \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)$$

$$\therefore\ \ln y = -x + \frac{1}{2}\left[\ln\left(1+x\right) - \ln\left(1-x\right)\right] \Rightarrow \therefore\ \ln y = -x + \frac{1}{2}\left[\ln\left(1+x\right) - \ln\left(1-x\right)\right]$$

Then differentiate w.r.to x : $\therefore\ \frac{y'}{y} = -1 + \frac{1}{2}\left[\frac{1}{1+x} - \frac{-1}{1-x}\right] = -1 + \frac{1}{2}\left[\frac{1}{1+x} + \frac{1}{1-x}\right]$

$$\therefore\ \frac{y'}{y} = -1 + \frac{1}{2}\left[\frac{\left(1-x\right)+\left(1+x\right)}{\left(1+x\right)\left(1-x\right)}\right] \Rightarrow \therefore\ \frac{y'}{y} = -1 + \frac{1}{2}\left[\frac{2}{1-x^2}\right] \Rightarrow \therefore\ \frac{y'}{y} = -1 + \frac{1}{1-x^2}$$

$$\therefore\ \frac{y'}{y} = \frac{-\left(1-x^2\right)}{1-x^2} + \frac{1}{1-x^2} = \frac{-1+x^2+1}{1-x^2} \Rightarrow \therefore\ \frac{y'}{y} = \frac{x^2}{1-x^2} \Rightarrow \boxed{\therefore\ y'\left(1-x^2\right) = x^2 y}$$

Example (14)

If $x = 5 + Sec^2 3\theta$ and $y = 1 - Tan 3\theta$, then find $\frac{d^2y}{dx^2}$ at $\theta = \frac{\pi}{4}$

Answer

Note This equations are called parametric equations

$$\because\ \frac{dx}{d\theta} = 2\left(Sec 3\theta\right)\left(3\ Sec 3\theta Tan 3\theta\right) \Rightarrow \boxed{\therefore\ \frac{dx}{d\theta} = 6\ Sec^2 3\theta\ Tan 3\theta} \quad \text{and} \quad \boxed{\because\ \frac{dy}{d\theta} = -3 Sec^2 3\theta}$$

$$\therefore\ \frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{-3 Sec^2 3\theta}{6\ Sec^2 3\theta\ Tan 3\theta} = \frac{-1}{2} Cot 3\theta \Rightarrow \therefore\ \frac{d^2y}{dx^2} = \frac{3}{2}\ Cosec^2\left(3\theta\right)\frac{d\theta}{dx}$$

$$\therefore\ \frac{d^2y}{dx^2} = \frac{Cosec^2\left(3\theta\right)}{4\ Sec^2\left(3\theta\right)\ Tan\left(3\theta\right)}$$

$$\text{At } x = \frac{\pi}{4} : \therefore\ \frac{d^2y}{dx^2} = \frac{Cosec^2\left(\frac{3\pi}{4}\right)}{4\ Sec^2\left(\frac{3\pi}{4}\right)\ Tan\left(\frac{3\pi}{4}\right)} = \frac{2}{\left(4\right)\left(2\right)\left(-1\right)} = \frac{-1}{4}$$
