The Exponential functions

Introduction

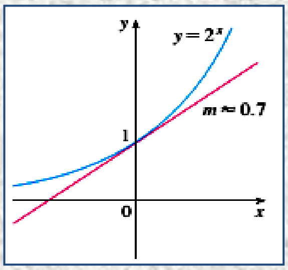
Last year, we have taken the exponential functions, and we have known that it has the form $f(x) = a^x$, where the base a is positive constant number except 1 . Exponential functions are useful for modeling many natural phenomena such as population growth $(a>1)$ or radioactivity decay $(a<1)$.

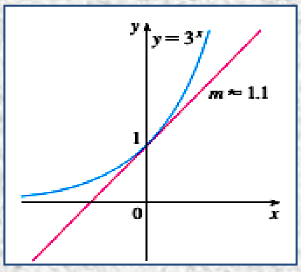
The natural Exponential function $\Rightarrow e^x$

For all possible bases of an exponential function $f(x) = a^x$, there is one that is most convenient for the purposes of calculus which is e^x.... why?
The choice of the base (a) is influenced by the way the graph $y = a^x$ crosses the y - axis .
e^x has a property that it is its own derivative, which means that it crosses the y - axis with slope 1.
The geometrical significance of this fact is that its y - coordinate (height) = its slope $\frac{dy}{dx}$.

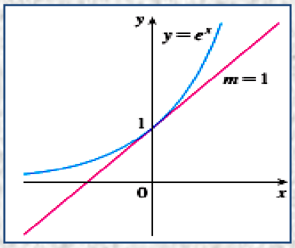
The magic number e

This notation was chosen by the Swiss mathematician Leonhard Euler in 1727, probably because it is the first letter of the word exponential.
If we want to find the slope of the tangent line to the curve $y = 2^x$ and $y = 3^x$ at point $(0,1)$ [see the fig.].
We found that when $y = 2^x$, its slope $m \simeq 0.7$.
Also we found that when $y = 3^x$, its slope $m \simeq 1.1$





I wonder when the slope of the tangent is equal to 1 .
In fact there is a number which is between 2^x and 3^x and its slope is 1. This number was $e = 2.718281828....$



As we said before e^x has a property that its $y = y'$ which will prove to you how this number came from.

$$e^x = y = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\times2}x^3 + ... \frac{1}{n(n-1)(n-2)}x^n$$

$$\frac{dy}{dx} = 1 + x + \frac{1}{2}x^2 + ... \frac{x^{n-1}}{(n-1)(n-2)} + \frac{x^n}{n(n-1)(n-2)}$$

$$\therefore \ e^x = y = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \frac{x^n}{n!}$$

Then by using taylor's series: $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$

By putting $x = 1$: $\therefore \ e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... \frac{1}{n!}$

Then by using taylor's series: $e = \sum_{n=0}^{\infty} \frac{1}{n!} = 2.718281828.....$

The derivative of natural exponential function $\Rightarrow e^x$

Rules

(1) If $y = e^x \Rightarrow \frac{dy}{dx} = e^x$ for $\forall \ x \in R$

(2) If $y = e^{g(x)} \Rightarrow \frac{dy}{dx} = e^{g(x)} \times g'(x)$

This means that if $y = e^{ax} \Rightarrow \frac{dy}{dx} = a e^{ax}$

Properties of e^x

Domain : R   Range : $]0, \infty[$ Type : one - one

Examples

Find the first derivative of each of the following functions:

(1) $y = e^{3x^2+1}$

Answer

$$\frac{dy}{dx} = e^{3x^2+1}\left[6x\right]$$

$$\frac{dy}{dx} = 6x\,e^{3x^2+1}$$

(2) $y = \sqrt{x}\;e^{x}$

Answer

$$\frac{dy}{dx} = \sqrt{x}\;e^{x} + e^{x}\left[\frac{1}{2\sqrt{x}}\right]$$

$$\frac{dy}{dx} = e^{x}\left[\sqrt{x} + \frac{1}{2\sqrt{x}}\right]$$

(3) $y = \dfrac{x}{e^{x}}$

Answer

$$\frac{dy}{dx} = \frac{e^{x} - x\,e^{x}}{e^{2x}} = \frac{e^{x}(1-x)}{e^{2x}}$$

$$\frac{dy}{dx} = \frac{1-x}{e^{x}}$$

(4) $y = x^2 e^{-3}$

Answer

$$\frac{dy}{dx} = \text{.......................}$$

Ans : $2x\,e^{-3}$

(5) $y = e^{Sec\,3\theta}$

Answer

$$\frac{dy}{dx} = e^{Sec\,3\theta}\left[3\,Sec\,3\theta\,Tan\,3\theta\right]$$

(6) $y = \sqrt{2 - e^{x^2}}$

Answer

$$\frac{dy}{dx} = \frac{-2x\,e^{x^2}}{2\sqrt{2 - e^{x^2}}} = \frac{-x\,e^{x^2}}{\sqrt{2 - e^{x^2}}}$$

(7) $f(x) = e^{x}\left(Tan\,x - x\right)$

Answer

$$f'(x) = e^{x}\left(Sec^2 x - 1\right) + \left(Tan\,x - x\right)e^{x}$$

$$f'(x) = e^{x}\left[Sec^2 x - 1 + Tan\,x - x\right]$$

$$f'(x) = e^{x}\left[Tan^2 x + Tan\,x - x\right]$$

(8) $f(\theta) = \dfrac{Cot\,\theta}{e^{\theta}}$

Answer

$$f'(\theta) = \frac{e^{\theta}\left(-Cosec^2\theta\right) - \left(Cot\,\theta\right)e^{\theta}}{\left(e^{\theta}\right)^2}$$

$$f'(\theta) = \frac{-e^{\theta}\left(Cosec^2\theta + Cot\,\theta\right)}{\left(e^{\theta}\right)^2} = \frac{-\left(Cosec^2\theta + Cot\,\theta\right)}{e^{\theta}}$$

(9) $f(x) = x\,e^{x}\,Cosec\,x$

Answer

$$f'(x) = x\,e^{x}\left[-Cosec\,x\,Cot\,x\right] + Cosec\,x\left[x\,e^{x} + e^{x}\right]$$

$$f'(x) = -x\,e^{x}\,Cosec\,x\,Cot\,x + x\,e^{x}\,Cosec\,x + e^{x}\,Cosec\,x$$

$$f'(x) = e^{x}\,Cosec\,x\left[-x\,Cot\,x + x + 1\right]$$

(10) $f(x) = Cosec\left(e^{x}\right)$

Answer

$$f'(x) = e^{x}\left[-Cosec\left(e^{x}\right)Cot\left(e^{x}\right)\right]$$

$$f'(x) = -e^{x}\;Cosec\left(e^{x}\right)Cot\left(e^{x}\right)$$

(11) $f(x) = e^{x\,Sin\,2x}$

Answer

$$f'(x) = e^{x\,Sin\,2x}\left[x\left(2\,Cos\,2x\right) + Sin\,2x\right]$$

$$f'(x) = e^{x\,Sin\,2x}\left[2x\,Cos\,2x + Sin\,2x\right]$$

(12) $f(x) = e^{k\,Tan\sqrt{x}}$ where k is constant

Answer

$$f'(x) = e^{k\,Tan\sqrt{x}}\left[k\;Sec^2\sqrt{x}\;\times\left(\frac{1}{2\sqrt{x}}\right)\right]$$

$$f'(x) = \frac{k\,e^{k\,Tan\sqrt{x}}\;Sec^2\sqrt{x}}{2\sqrt{x}}$$

Example (13)

Find the first derivative of $f\left(x\right)=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$

Answer

$$f'\left(x\right)=\frac{\left(e^{x}+e^{-x}\right)\left(e^{x}+e^{-x}\right)-\left(e^{x}-e^{-x}\right)\left(e^{x}-e^{-x}\right)}{\left(e^{x}+e^{-x}\right)^{2}}=\frac{\left(e^{2x}+e^{0}+e^{0}+e^{-2x}\right)-\left[e^{2x}-e^{0}-e^{0}+e^{-2x}\right]}{\left(e^{x}+e^{-x}\right)^{2}}$$

$$f'\left(x\right)=\frac{\cancel{e^{2x}}+1+1+\cancel{e^{-2x}}-\cancel{e^{2x}}+1+1-\cancel{e^{-2x}}}{\left(e^{x}+e^{-x}\right)^{2}}=\frac{4}{\left(e^{x}+e^{-x}\right)^{2}}$$

Example (14)

Find the first derivative of $f\left(x\right)=\dfrac{1-xe^{x}}{x+e^{x}}$

Answer

$$f'\left(x\right)=\frac{\left(x+e^{x}\right)\left(-xe^{x}-e^{x}\right)-\left(1-xe^{x}\right)\left(1+e^{x}\right)}{\left(x+e^{x}\right)^{2}}=\frac{\left(-x^{2}e^{x}-xe^{x}-xe^{2x}-e^{2x}\right)-\left[1+e^{x}-xe^{x}-xe^{2x}\right]}{\left(x+e^{x}\right)^{2}}$$

$$f'\left(x\right)=\frac{-x^{2}e^{x}-\cancel{xe^{x}}-\cancel{xe^{2x}}-e^{2x}-1-e^{x}+\cancel{xe^{x}}+\cancel{xe^{2x}}}{\left(x+e^{x}\right)^{2}}=\frac{-x^{2}e^{x}-e^{2x}-1-e^{x}}{\left(x+e^{x}\right)^{2}}$$

Example (15)

Find the first derivative of $f\left(x\right)=Cos\left(\dfrac{1-e^{2x}}{1+e^{2x}}\right)$

Answer

$$f'\left(x\right)=-Sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\times\frac{\left(1+e^{2x}\right)\left(-2e^{2x}\right)-\left(1-e^{2x}\right)\left(2e^{2x}\right)}{\left(1+e^{2x}\right)^{2}}$$

$$f'\left(x\right)=-Sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\times\frac{\left(-2e^{2x}-2e^{4x}\right)-\left(2e^{2x}-2e^{4x}\right)}{\left(1+e^{2x}\right)^{2}}$$

$$f'\left(x\right)=-Sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)\times\frac{-2e^{2x}-\cancel{2e^{4x}}-2e^{2x}+\cancel{2e^{4x}}}{\left(1+e^{2x}\right)^{2}}=\frac{4e^{2x}}{\left(1+e^{2x}\right)^{2}}Sin\left(\frac{1-e^{2x}}{1+e^{2x}}\right)$$

Example (16)

Find the first derivative of $y=Sin^{2}\left(e^{Sin^{2}x}\right)$

Answer

$$y'=2\left[Sin\left(e^{Sin^{2}x}\right)\right]\left[Cos\left(e^{Sin^{2}x}\right)\times\left(e^{Sin^{2}x}\times 2\,Sin\,x\,Cos\,x\right)\right]$$

$$y'=2\,Sin\left(e^{Sin^{2}x}\right)Cos\left(e^{Sin^{2}x}\right)\times\left[e^{Sin^{2}x}\times Sin\,2x\right]=e^{Sin^{2}x}\,Sin\left(2x\right)\,Sin\left(2e^{Sin^{2}x}\right)$$

The Logarithmic functions

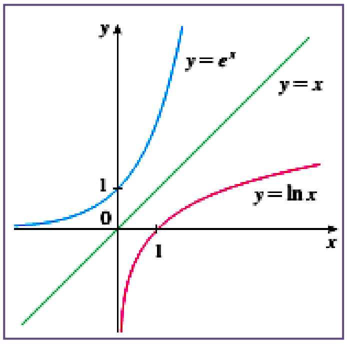
Introduction

Last year, we have taken the logarithmic functions, and we have known that it has the form $f(x)=\log_a x$, where the base a is positive constant number except 1 .
And we have taken some rules of logarithms :

Rule – 1 If $\log_a x=y \Rightarrow \therefore a^y=x$	Rule – 2 $\log_a x^{n}=n\log_a x$
Rule – 3 $\log_a xy=\log_a x+\log_a y$	Rule – 4 $\log_a \frac{x}{y}=\log_a x-\log_a y$
Rule – 5 $\log x \xrightarrow{Means} \log_{10} x$	Rule – 6 $\log_a a=1$

The natural logarithmic function $\Rightarrow \ln x$

It comes from the latine name "logarithmus naturali" and it is denoted by $\ln x$ $(\log_e x)$ where $\ln x$ is inverse to e^x (see the fig)



Properties of ln

Inverse of e^x	Domain : R^+	Range : R	Type : one - one

Remark: $\ln x$ crosses the x - axis with slope 1

Rules

Rule – 1 If $\ln_e x=y \Rightarrow \therefore e^y=x$	Rule – 2 $\ln x=\ln_e x$, where $x\in R^+$
Rule – 3 $\ln e=1$ (as they are inverse)	Rule – 4 $\ln x^{n}=n\ln x$
Rule – 5 $\log_a x=\frac{\ln x}{\ln a}$ (from log to ln)	Rule – 6 $\ln_e x=\frac{1}{\log_x e}$ (from ln to log)
Rule – 7 $\ln xy=\ln x+\ln y$	Rule – 8 $\ln\frac{x}{y}=\ln x-\ln y$
Rule – 9 when power is unknown (ln both sides)	Rule – 10 $\ln e^x=e^{\ln x}=x$

Sample Examples

(1) **If** $\ln x = 0.3010$, **find** x **Answer** $\because e^{0.3010} = x \Rightarrow \boxed{\therefore\ x \simeq 1.35}$	(2) Find the value of x in : $e^{\ln(x+1)} = 4$ **Answer** $\because x+1 = 4 \Rightarrow \boxed{\therefore\ x = 3}$
(3) If $\ln(x+1) + \ln(x-1) = 1$, find x **Answer** $\ln_e(x+1)(x-1) = 1 \Rightarrow \therefore e = x^2 - 1$ $\therefore x^2 = e+1 \Rightarrow \boxed{\therefore\ x = \sqrt{e+1}}$	(4) If $\ln(x^2-4) - \ln(x-2) = 0$, find x **Answer** $\ln\frac{(x^2-4)}{(x-2)} = 0 \Rightarrow \therefore \ln\frac{(x-2)(x+2)}{(x-2)} = 0$ $\ln(x+2) = 0 \Rightarrow \therefore e^0 = x+2 \Rightarrow \therefore 1 = x+2$ $\boxed{x = -1 \text{ refused}} \Rightarrow S.S. = \varnothing$

(5) Find the S.S. of x in : $2^x = 100$ to the nearest two decimal places

Answer

$\because$ The power is unknown, then we can ln both sides

$\therefore \ln 2^x = \ln 100 \Rightarrow \therefore x \ln 2 = \ln 100 \Rightarrow x = \frac{\ln 100}{\ln 2} = \log_2 100 = 6.64$

The derivative of the Exponential and logarithmic functions

The derivative of $\boxed{a^x}$	The derivative of $\boxed{\ln x}$	The derivative of $\boxed{\log_a x}$
Rules (1) If $y = a^x$, $a \in R^+ - \{1\}$ Then $\boxed{\frac{dy}{dx} = a^x \ln a}$ (2) If $y = a^{bx}$, $a \in R^+ - \{1\}$ Then $\boxed{\frac{dy}{dx} = \left[a^{bx} \ln a\right] \times b}$ **Generally** If $y = a^{g(x)}$, $a \in R^+ - \{1\}$ $\boxed{\therefore \frac{dy}{dx} = \left[a^{g(x)} \ln a\right] \times g'(x)}$	**Rules** (1) If $y = \ln x$ or $y = \ln\lvert x\rvert$ Then $\boxed{\frac{dy}{dx} = \frac{1}{x}}$ (2) If $y = \ln ax$ Then $\boxed{\frac{dy}{dx} = \frac{a}{ax}}$ **Generally** If $y = \ln g(x)$ $\boxed{\therefore \frac{dy}{dx} = \frac{g'(x)}{g(x)} = \frac{\text{derivative}}{\text{itself}}}$	**Rules** (1) If $y = \log_a x$ Then $\boxed{\frac{dy}{dx} = \frac{1}{x \ln a}}$ (2) If $y = \log_a bx$ Then $\boxed{\frac{dy}{dx} = \frac{b}{bx \ln a}}$ **Generally** If $y = \log_a g(x)$ $\boxed{\frac{dy}{dx} = \frac{g'(x)}{g(x) \ln a} = \frac{\text{derivative}}{\text{itself} \times \ln \text{base}}}$

Examples

Find the first derivative of each of the following functions:

Example (1)

$$f(x) = 10^{1-x^2}$$

Answer

Note use the rule of exponentials when the **base is constant** and **power is variable**

$$\text{Then } f'(x) = (\text{itself}) \times [\ln(\text{base})] \times [\text{derivative the power}]$$

$$\therefore\ f'(x) = 10^{1-x^2} \times [\ln 10][-2x] = -2x\left[10^{1-x^2}\right][\ln 10]$$

Example (2)

$$f(x) = (5+x)\,6^x$$

Answer

Note use the rule of exponentials when the **base is constant** and **power is variable**

$$\text{Then } f'(x) = (\text{itself}) \times [\ln(\text{base})] \times [\text{derivative the power}]$$

$$\therefore\ f'(x) = (5+x) \times \left[6^x(\ln 6)(1)\right] + 6^x \times 1 = (5+x)\ 6^x \ln 6 + 6^x$$

Example (3)

$$y = e^{\sin x} \times 2^{-5x}$$

Answer

Note use the rule of exponentials when the **base is constant** and **power is variable**

$$\frac{dy}{dx} = e^{\sin x}\left[2^{-5x}\right][\ln 2][-5] + 2^{-5x}\left[e^{\sin x}\right][\cos x] = -5\ln 2\left(e^{\sin x}\right)\left(2^{-5x}\right) + 2^{-5x}\left[e^{\sin x}\right][\cos x]$$

$$\frac{dy}{dx} = 2^{-5x}\ e^{\sin x}\left[\cos x - 5\ln 2\right]$$

Example (4)

$$y = 2^{\sin(\pi x)}$$

Answer

....

....

....

.... $\text{Ans}: \left(\pi\ \ln(2)\cos \pi x\right)\left(2^{\sin(\pi x)}\right)$

$(5)\ y = 3x + \ln x$

Answer

$$\frac{dy}{dx} = 3 + \frac{1}{x}$$

$(6)\ y = \ln\left(2x^3 + 9\right)$

Answer

$$\frac{dy}{dx} = \frac{6x^2}{2x^3 + 9}$$

$(7)\ y = x^4 \ln\left(x^3\right)$

Answer

$$\frac{dy}{dx} = x^4\left[\frac{3x^2}{x^3}\right] + \ln\left(x^3\right)\left[4x^3\right] = 3x^3 + 4x^3 \ln\left(x^3\right)$$

$(8)\ y = \dfrac{\ln x - 1}{\ln x + 1}$

Answer

$$\frac{dy}{dx} = \frac{(\ln x + 1)\left(\frac{1}{x}\right) - (\ln x - 1)\left(\frac{1}{x}\right)}{(\ln x + 1)^2}$$

$$\frac{dy}{dx} = \frac{\frac{1}{x}\ln x + \frac{1}{x} - \frac{1}{x}\ln x + \frac{1}{x}}{(\ln x + 1)^2} = \frac{2}{x(\ln x + 1)^2}$$

$(9)\ y = \sqrt{\ln x}$

Answer

$$\frac{dy}{dx} = \frac{\frac{1}{x}}{2\sqrt{\ln x}} = \frac{1}{2x\sqrt{\ln x}}$$

$(10)\ y = \ln(\mathrm{Cosec}\, x)$

Answer

....

....

....

Ans : $-\mathrm{Cot}\, x$

$(11)\ y = \log_5(3x - 2)$

Answer

$$\frac{dy}{dx} = \frac{3}{(3x - 2)\ln 5}$$

$(12)\ y = \log(2x - 3)^2$

Answer

$$\frac{dy}{dx} = \frac{2(2x - 3)\times 2}{(2x - 3)^2 \ln 10} = \frac{4}{(2x - 3)\ln 10}$$

$(13)\ y = \log(2 + \mathrm{Sin}\, x)$

Answer

$$\frac{dy}{dx} = \frac{\mathrm{Cos}\, x}{(2 + \mathrm{Sin}\, x)\ln 10}$$

Mixed Examples

Find the first derivative of each of the following functions:

Example (1)

$f(x) = 3^{3x^2 - 5x + 2}$

Answer

Note use the rule of exponentials when the **base is constant** and **power is variable**

Then $f'(x) = (\text{itself}) \times [\ln(\text{base})] \times [\text{derivative the power}]$

$$\therefore\ f'(x) = 3^{x^2 - 5x + 2} \times [\ln 3][6x - 5] = 3^{x^2 - 5x + 2}(6x - 5)\ln 3$$

Example (2)

$y = 2^{\mathrm{Sec}^2 x}$

Answer

Note use the rule of exponentials when the **base is constant** and **power is variable**

$$\frac{dy}{dx} = 2^{\mathrm{Sec}^2 x}[\ln 2]\left[2(\mathrm{Sec}\, x)(\mathrm{Sec}\, x \,\mathrm{Tan}\, x)\right] = 2\,\mathrm{Sec}^2 x\,\mathrm{Tan}\, x\left[2^{\mathrm{Sec}^2 x}\ln 2\right]$$

Example (3)

$$y=\left(3^{2x-1}\right)^{-2}$$

Answer

$$\frac{dy}{dx}=-2\left(3^{2x-1}\right)^{-3}\left[3^{2x-1}\ln(3)\times 2\right]=-4\left(3^{2x-1}\right)\left(3^{2x-1}\right)^{-3}\ln(3)=-4\left(3^{2x-1}\right)^{-2}\ln(3)$$

Example (4)

$$y=e^{x}\ln\left(x^{2}+1\right)$$

Answer

$$\because\ \frac{dy}{dx}=e^{x}\left[\frac{2x}{x^{2}+1}\right]+\ln\left(x^{2}+1\right)\times\left[e^{x}\right]=e^{x}\left[\frac{2x}{x^{2}+1}+\ln\left(x^{2}+1\right)\right]$$

Example (5)

$$y=\log_{5}\left(x\,e^{x}\right)$$

Answer

$$\because\ \frac{dy}{dx}=\frac{xe^{x}+e^{x}}{xe^{x}\left[\ln 5\right]}=\frac{e^{x}\left(x+1\right)}{xe^{x}\ln 5}=\frac{x+1}{x\ln 5}$$

Example (6)

$$y=2^{(3)^{x^2}}$$

Answer

$$\frac{dy}{dx}=2^{(3)^{x^2}}\left[\ln(2)\right]\left[3^{x^2}\right]\left[\ln(3)\right](2x)=2x\left[2^{(3)^{x^2}}\ln(2)\right]\left[3^{x^2}\ln(3)\right]$$

Example (7)

$$y=\ln\left(x+\sqrt{x^{2}-1}\right)$$

Answer

$$\because\ \frac{dy}{dx}=\frac{1+\dfrac{2x}{2\sqrt{x^{2}-1}}}{x+\sqrt{x^{2}-1}}=\frac{\dfrac{\sqrt{x^{2}-1}}{\sqrt{x^{2}-1}}+\dfrac{x}{\sqrt{x^{2}-1}}}{x+\sqrt{x^{2}-1}}=\frac{\dfrac{\sqrt{x^{2}-1}+x}{\sqrt{x^{2}-1}}}{x+\sqrt{x^{2}-1}}=\frac{\left(\sqrt{x^{2}-1}+x\right)}{\sqrt{x^{2}-1}\left(x+\sqrt{x^{2}-1}\right)}=\frac{1}{\sqrt{x^{2}-1}}$$

Example (8)

$$y = \ln\left(x\sqrt{x^2-1}\right)$$

Answer

$$\because\ y = \ln x + \ln\left(\sqrt{x^2-1}\right) \Rightarrow \therefore\ \frac{dy}{dx} = \frac{1}{x} + \frac{\frac{2x}{2\sqrt{x^2-1}}}{\sqrt{x^2-1}} = \frac{1}{x} + \frac{x}{x^2-1} = \frac{x^2-1+x^2}{x\left(x^2-1\right)} = \frac{2x^2-1}{x\left(x^2-1\right)}$$

Note you can differentiate directly, but it will be longer

Example (9)

$$y = \ln\frac{(2x+1)^5}{\sqrt{x^2+1}}$$

Answer

$$\because\ y = \ln(2x+1)^5 - \ln\left(\sqrt{x^2+1}\right) = 5\ln(2x+1) - \ln\left(\sqrt{x^2+1}\right)$$

$$\therefore\ \frac{dy}{dx} = \frac{5\times 2}{2x+1} - \frac{\frac{2x}{2\left(\sqrt{x^2+1}\right)}}{\left(\sqrt{x^2+1}\right)} = \frac{10}{2x+1} - \frac{x}{x^2+1} = \frac{10x^2+10-2x^2-x}{(2x+1)\left(x^2+1\right)} = \frac{8x^2-x+10}{(2x+1)\left(x^2+1\right)}$$

Note you can differentiate directly, but it will be longer

Example (10)

If $\frac{dy}{dx} = 0$ in the function $y = x^3 \ln x$, then find the value of x $(\text{for } x > 0)$

Answer

$$\frac{dy}{dx} = x^3 \times \left[\frac{1}{x}\right] + \ln x\left[3x^2\right] = x^2 + 3x^2\ln x = x^2\left[1+3\ln x\right]$$

When $\frac{dy}{dx} = 0 \Rightarrow \therefore\ x^2\left[1+3\ln x\right] = 0 \Rightarrow \therefore\ x^2 = 0$ (refused) or $1 + 3\ln x = 0$

Then $3\ln x = -1 \Rightarrow \therefore\ \ln x = \frac{-1}{3} \Rightarrow \therefore\ e^{\frac{-1}{3}} = x \Rightarrow \boxed{\therefore\ x = \frac{1}{\sqrt[3]{e}}}$

Example (11)

If $\frac{dy}{dx}=0$ in the function $y=\frac{1}{x^2}\ln x^2$, then find the value of x $\left(\text{for } x\neq 0\right)$

Answer

$$\text{Let } y=\frac{\ln x^2}{x^2} \Rightarrow \therefore \frac{dy}{dx}=\frac{x^2\left[\frac{2x}{x^2}\right]-\ln x^2\left[2x\right]}{x^4}=\frac{2x-2x\ln x^2}{x^4}=\frac{2x\left[1-\ln x^2\right]}{x^4}=\frac{2\left[1-\ln x^2\right]}{x^3}$$

$$\text{When } \frac{dy}{dx}=0 \Rightarrow \therefore 2x\left[1-\ln x^2\right]=0 \Rightarrow \therefore 2x=0 \Rightarrow \therefore x=0 \ (\text{refused})$$

$$\text{Or } 1-\ln x^2=0 \Rightarrow \therefore \ln x^2=1 \Rightarrow \therefore e=x^2 \Rightarrow \boxed{\therefore x=\sqrt{e} \text{ or } x=-\sqrt{e}}$$

Example (12)

If $y=\log\left(\sqrt{x}\right)^{2x}$, then prove that $\frac{dy}{dx}=\log\left(xe\right)$

Answer

$$\text{Let } y=2x\log\left(\sqrt{x}\right) \Rightarrow \therefore \frac{dy}{dx}=2x\times\left[\frac{\frac{1}{2\sqrt{x}}}{\sqrt{x}\ln 10}\right]+2\log\left(x\right)^{\frac{1}{2}}$$

$$\therefore \frac{dy}{dx}=2x\times\left[\frac{1}{2x\ln 10}\right]+2\times\frac{1}{2}\log\left(x\right) \Rightarrow \boxed{\therefore \frac{dy}{dx}=\frac{1}{\ln 10}+\log\left(x\right)}$$

Note : in order to change $\ln x \xrightarrow{To} \log x$

Rule $\ln 10=\frac{1}{\log_{10} e}$

$$\text{From } (1): \therefore \frac{dy}{dx}=\frac{1}{\left(\frac{1}{\log_{10} e}\right)}+\log\left(x\right)=\log e+\log\left(x\right)=\log\left(xe\right)$$

Example (13)

Find the first derivative of the equation: $y=\log_2\left(e^{-x}\cos\left(\pi x\right)\right)$

Answer

$$\text{Let } y=\log_2\left(e^{-x}\right)+\log_2\left(\cos\left(\pi x\right)\right) \Rightarrow \therefore y'=\frac{-e^{-x}}{e^{-x}\ln 2}+\frac{-\pi\sin\left(\pi x\right)}{\cos\left(\pi x\right)\ln 2}=\frac{-1}{\ln 2}+\frac{-\pi\tan\left(\pi x\right)}{\ln 2}$$

$$\boxed{\therefore y'=\frac{-\left(1+\pi\tan\left(\pi x\right)\right)}{\ln 2}} \text{ or } y'=\frac{-\left(1+\pi\tan\left(\pi x\right)\right)}{\frac{1}{\log_2 e}} \Rightarrow \boxed{y'=-\log_2 e\left(1+\pi\tan\left(\pi x\right)\right)}$$

Example (14)

Find the first derivative of the equation: $y = \dfrac{e^{3x}}{\log x}$

Answer

$$\because\ y' = \frac{\log x\left[3e^{3x}\right] - e^{3x}\left[\dfrac{1}{x\ln 10}\right]}{(\log x)^2} = \frac{3e^{3x}\log x - \dfrac{e^{3x}}{x\ln 10}}{(\log x)^2} ---(1)$$

Note when $\ln x$ and $\log x$ appear at the same problem , it is preferred to make the problem in one shape (either $\ln x$ or $\log x$)

Note : in order to change $\ln x \xrightarrow{To} \log x$ $\ln 10 = \dfrac{1}{\log e}$

$$\text{Substitute in } (1):\ y' = \frac{3e^{3x}\log x - \dfrac{e^{3x}\log e}{x}}{(\log x)^2} = \frac{\dfrac{3xe^{3x}\log x}{x} - \dfrac{e^{3x}\log e}{x}}{(\log x)^2} = \frac{e^{3x}\left[3x\log x - \log e\right]}{x(\log x)^2}$$

$$\therefore\ y' = e^{3x}\left[\frac{3}{\log x} - \frac{\log e}{x(\log x)^2}\right]$$

Another method Note : in order to change $\log x \xrightarrow{To} \ln x$ $\log_{10} x = \dfrac{\ln x}{\ln 10}$

$$\therefore\ y' = \frac{3e^{3x}\dfrac{\ln x}{\ln 10} - \dfrac{e^{3x}}{x\ln 10}}{\left(\dfrac{\ln x}{\ln 10}\right)^2} \Rightarrow \therefore\ y' = \frac{\dfrac{3e^{3x}\ln x}{\ln 10} - \dfrac{e^{3x}}{x\ln 10}}{\dfrac{(\ln x)^2}{(\ln 10)^2}} \Rightarrow \therefore\ y' = \frac{\dfrac{3x\,e^{3x}\ln x}{x\ln 10} - \dfrac{e^{3x}}{x\ln 10}}{\dfrac{(\ln x)^2}{(\ln 10)^2}}$$

$$\therefore\ y' = \frac{\dfrac{3x\,e^{3x}\ln x - e^{3x}}{x\ln 10}}{\dfrac{(\ln x)^2}{(\ln 10)^2}} \therefore\ y' = \frac{(\ln 10)^2\left(3x\,e^{3x}\ln x - e^{3x}\right)}{x\ln 10\,(\ln x)^2}$$

$$\therefore\ y' = \frac{e^{3x}\ln 10(3x\ln x - 1)}{x(\ln x)^2} = e^{3x}\ln 10\left[\frac{3}{\ln x} - \frac{1}{x(\ln x)^2}\right]$$

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4 LESSON

Implicit differentiation

Introduction

From our previous study, we have taken the explicit differentiation.

And we knew that: If y is a differentiable function with respect to x, then the derivative of:

$$y = x^n \quad \textbf{is} \quad y' = \frac{dy}{dx} = n\,x^{n-1}$$

So, an **Explicit** function is one which is given in terms of the independent variable.

Shape of Explicit function like $y = x^2 + 3x - 8$

On the other hand an **Implicit** function are usually given in terms of both dependent and independent variables.

Shape of Implicit function like $x\,y^2 + x^2 y - 3x + 8 = 0$

Note: Sometimes Implicit functions may be Explicit like: $y - x = 3$

So, how can we differentiate Implicit functions? let's ask another question

Does $\frac{d}{dx} y^3 = 3y^2$? —The answer is→ No ..... why ?!!!!

Because when we differentiate with respect to x and the answer doesn't include x.

Then we must add $\frac{dy}{dx}$ in order to respect that x

So $\frac{d}{dx} y^3 \Rightarrow 3y^2 \frac{dy}{dx}$ ⟶ This is the right answer

Some examples on derivatives with respect to x

(1) $\frac{d}{dx} x\,y = x\frac{dy}{dx} + y(1) = x\frac{dy}{dx} + y$

(2) $\frac{d}{dx} 2x^2 y^3 = 2x^2\left(3y^2\right)\frac{dy}{dx} + y^3(4x) = 6x^2y^2\frac{dy}{dx} + 4x\,y^3$

(3) $\frac{d}{dx}\left(\frac{x}{y}\right) = \dfrac{y - x\frac{dy}{dx}}{y^2}$

(4) $\frac{d}{dx}\left(x^2 - 3x + 2\right)^6 \Rightarrow 6\left(x^2 - 3x + 2\right)^5(2x - 3) = 12x - 18\left(x^2 - 3x + 2\right)^3$

(5) $\frac{d}{dx}\left(1 + 2x\sqrt{y}\right)^7 = 7\left(1 + 2x\,y^{\frac{1}{2}}\right)^6\left(2x\left(\frac{1}{2}\right)y^{\frac{-1}{2}}\frac{dy}{dx} + y^{\frac{1}{2}}(2)\right) = 7\left(1 + 2x\,y^{\frac{1}{2}}\right)^6\left(\frac{x}{\sqrt{y}}\frac{dy}{dx} - 2\sqrt{y}\right)$

Examples

Find the first derivative in each of the following :

$(1)\ x^2+4xy-3y^2-2x=5$

Answer

Differentiate with respect to x :

$$2x+4x\times\frac{dy}{dx}+4y-6y\frac{dy}{dx}-2=0\ \text{" Divide by 2"}\ \Rightarrow\ x+2x\times\frac{dy}{dx}+2y-3y\frac{dy}{dx}-1=0$$

$$\frac{dy}{dx}\left[2x-3y\right]=1-2y-x\ \Rightarrow\ \frac{dy}{dx}=\frac{1-2y-x}{2x-3y}$$

$(2)\ y^3+2x^2y-3xy^2=0$

Answer

Differentiate with respect to x :

$$3y^2\times\frac{dy}{dx}+2x^2\frac{dy}{dx}+4xy-\left[3x(2y)\frac{dy}{dx}+3y^2\right]=0\ \Rightarrow 3y^2\frac{dy}{dx}+2x^2\frac{dy}{dx}+4xy-6xy\frac{dy}{dx}-3y^2=0$$

$$\frac{dy}{dx}\left[3y^2+2x^2-6xy\right]=3y^2+4xy\ \Rightarrow\ \frac{dy}{dx}=\frac{3y^2-4xy}{3y^2+2x^2-6xy}$$

$(3)\ \text{Sin}\,3y-\text{Cos}\,2x=0$

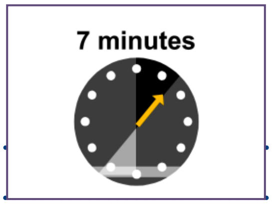
Answer

Differentiate with respect to x :

$\text{Sin}\,3y-\text{Cos}\,2x=0$

$$3\frac{dy}{dx}\text{Cos}\,3y+2\text{Sin}\,2x=0\ \Rightarrow\ 3\frac{dy}{dx}\text{Cos}\,3y=-2\text{Sin}\,2x\ \Rightarrow\ \frac{dy}{dx}=\frac{-2\text{Sin}\,2x}{3\text{Cos}\,3y}$$

$(4)\ x\,\text{Cos}\,y+y\,\text{Sin}\,2x=1$



Answer

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$$Ans:\ \frac{dy}{dx}=\frac{-\text{Cos}\,y-2y\,\text{Cos}\,2x}{\text{Sin}\,2x-x\,\text{Sin}\,y}$$

Example (5)

If $\sin y - \cos 2x + \tan 4x = 0$, then find $\frac{dy}{dx}$ at $x = \frac{\pi}{4}$

Answer

$$Tan4x = \frac{Sin4x}{Cos4x} \Rightarrow \therefore \frac{d}{dx}Tan4x = 4Sec^2 4x = \frac{4}{Cos^2 4x}$$

So , by differentiate with respect to x: $\cos y \frac{dy}{dx} + 2\sin 2x + 4\sec^2 4x = 0 - - - (1)$

When $x = \frac{\pi}{4} \Rightarrow \sin y - \cos 90^o + \tan 180^o = 0 \Rightarrow \therefore \sin y = 0 \Rightarrow \boxed{\therefore y = 0}$

Then by substitute in (1): $\cos(0)\frac{dy}{dx} + 2\sin\left(\frac{\pi}{2}\right) + 4\sec^2\pi = 0 \Rightarrow \frac{dy}{dx} + 2 + 4 = 0 \Rightarrow \boxed{\frac{dy}{dx} = -6}$

Example (6)

If $e^y \cos x = 1 + \sin xy$, then find $\frac{dy}{dx}$

Answer

By differentiating w. r. t x: $e^y[-\sin x] + \cos x\left[e^y\right]\frac{dy}{dx} = \cos(xy)\left[x\frac{dy}{dx} + y\right]$

$$\therefore -e^y \sin x + e^y \frac{dy}{dx}\cos x = x\frac{dy}{dx}\cos(xy) + y\cos(xy)$$

$$\therefore e^y \frac{dy}{dx}\cos x - x\frac{dy}{dx}\cos(xy) = y\cos(xy) + e^y \sin x$$

$$\therefore \frac{dy}{dx}\left[e^y \cos x - x\cos(xy)\right] = y\cos(xy) + e^y \sin x \Rightarrow \boxed{\therefore \frac{dy}{dx} = \frac{y\cos(xy) + e^y \sin x}{e^y \cos x - x\cos(xy)}}$$

Example (7)

If $e^{xy} = x^2 + y$, then prove that: $\left(xe^{xy} - 1\right)y' = 2x - ye^{xy}$

Answer

By differentiating w. r. t x: $e^{xy}[xy' + y] = 2x + y' \Rightarrow \therefore e^{xy}xy' + e^{xy}y = 2x + y'$

$$\therefore e^{xy}xy' - y' = 2x - e^{xy}y \Rightarrow \therefore y'\left(e^{xy}x - 1\right) = 2x - e^{xy}y$$

Example (8)

If $y = \ln\left(x^2 + y^2\right)$, then find $\frac{dy}{dx}$.

Answer

By differentiating w. r. t x: $\therefore \frac{dy}{dx} = \frac{2x + 2y\frac{dy}{dx}}{x^2 + y^2} \Rightarrow \therefore \frac{dy}{dx}x^2 + \frac{dy}{dx}y^2 = 2x + 2y\frac{dy}{dx}$

$$\therefore \frac{dy}{dx}x^2 + \frac{dy}{dx}y^2 - 2y\frac{dy}{dx} = 2x \Rightarrow \therefore \frac{dy}{dx}\left[x^2 + y^2 - 2y\right] = 2x \Rightarrow \therefore \frac{dy}{dx} = \frac{2x}{x^2 + y^2 - 2y}$$

Example (9)

If $e^{x/y} = x - y$, then find $\dfrac{dy}{dx}$

Answer

By differentiating w. r. t x: $e^{x/y}\left[\dfrac{y - x\dfrac{dy}{dx}}{y^2}\right] = 1 - \dfrac{dy}{dx}$ $\left(\text{multiply both by } y^2\right)$

$$\therefore\ e^{x/y}\left(y - x\frac{dy}{dx}\right) = y^2 - y^2\frac{dy}{dx} \Rightarrow \therefore\ y e^{x/y} - x\, e^{x/y}\frac{dy}{dx} = y^2 - y^2\frac{dy}{dx}$$

$$\therefore\ y^2\frac{dy}{dx} - x\, e^{x/y}\frac{dy}{dx} = y^2 - y e^{x/y} \Rightarrow \frac{dy}{dx}\left[y^2 - x\, e^{x/y}\right] = y^2 - y e^{x/y} \Rightarrow \boxed{\therefore\ \frac{dy}{dx} = \frac{y^2 - y e^{x/y}}{y^2 - x\, e^{x/y}}}$$

Example (10)

If $x\, Cosec\ y = y\ Cot\, x$, then find $\dfrac{dx}{dy}$

Answer

By differentiating w. r. to y: $\therefore\ x\left[-Cosec\, y\ Cot\, y\right] + Cosec\, y\left[\dfrac{dx}{dy}\right] = y\left[-Cosec^2 x\right]\left[\dfrac{dx}{dy}\right] + Cot\, x$

$$\therefore\ -x Cosec\, y\ Cot\, y + \frac{dx}{dy} Cosec\, y = -y\ \frac{dx}{dy} Cosec^2 x + Cot\, x$$

$$\therefore\ \frac{dx}{dy} Cosec\, y + y\ \frac{dx}{dy} Cosec^2 x = Cot\, x + x Cosec\, y\ Cot\, y$$

$$\therefore\ \frac{dx}{dy}\left[Cosec\, y + y\ Cosec^2 x\right] = Cot\, x + x Cosec\, y\ Cot\, y \Rightarrow \boxed{\therefore\ \frac{dx}{dy} = \frac{Cot\, x + x Cosec\, y\ Cot\, y}{Cosec\, y + y\ Cosec^2 x}}$$

Example (11)

If $x\, e^{\frac{-1}{2}y} + y\, e^{\frac{-1}{2}x} = 2$, then find $\dfrac{dy}{dx}$ at $x = 0$

Answer

By differentiating w. r. t x: $x\, e^{\frac{-1}{2}y}\left[\dfrac{-1}{2}\dfrac{dy}{dx}\right] + e^{\frac{-1}{2}y} + y\, e^{\frac{-1}{2}x}\left[\dfrac{-1}{2}\right] + e^{\frac{-1}{2}x}\left[\dfrac{dy}{dx}\right] = 0$

$$\therefore\ \frac{dy}{dx} e^{\frac{-1}{2}x} - \frac{1}{2}x\, e^{\frac{-1}{2}y}\frac{dy}{dx} = \frac{1}{2}y\, e^{\frac{-1}{2}x} - e^{\frac{-1}{2}y} \Rightarrow \therefore\ \frac{dy}{dx}\left[e^{\frac{-1}{2}x} - \frac{1}{2}x e^{\frac{-1}{2}y}\right] = \frac{1}{2}y\, e^{\frac{-1}{2}x} - e^{\frac{-1}{2}y}$$

$\boxed{\therefore\ \dfrac{dy}{dx} = \dfrac{\frac{1}{2}y\, e^{\frac{-1}{2}x} - e^{\frac{-1}{2}y}}{e^{\frac{-1}{2}x} - \frac{1}{2}x\, e^{\frac{-1}{2}y}}}$, then substitute in the original function by $x = 0 \Rightarrow \boxed{\therefore\ y = 2}$

Then at $(0,2) \Rightarrow \therefore\ \dfrac{dy}{dx} = \dfrac{\frac{1}{2}(2)\, e^{0} - e^{-1}}{e^{0} - 0} = 1 - \dfrac{1}{e} = \dfrac{e-1}{e}$

Lesson 5: Complicated differentiation

Introduction

To differentiate a complicated function, it may take a long time to solve it by the traditional way of rules we have taken before, so solving this kind of differentiation which involves complicated products, quotients, or powers can often be simplified by taking logarithms.

You should distinguish carefully between the Power Rule $\frac{d}{dx}\left(x^n\right) = n\,x^{n-1}$*, where the base is variable and the exponent is constant, and the rule for differentiating exponential functions* $\frac{d}{dx}\left(a^x\right) = a^x \ln a$ *, where the base is constant and the exponent is variable.*

In general there are four cases for powers and bases

(1) Constant **base**, Constant **power** $\Rightarrow \frac{d}{dx}\left(a^b\right) = 0$

(2) Variable **base**, Constant **power** $\Rightarrow \frac{d}{dx}\left[f(x)\right]^b = b\left[f(x)\right]^{b-1} \times f'(x)$

(3) Constant **base**, Variable **power** $\Rightarrow \frac{d}{dx}\left[a\right]^{f(x)} = \left[a\right]^{f(x)} \ln(a) \times f'(x)$

(4) Variable **base**, Variable **power** $\Rightarrow \frac{d}{dx}\left[f(x)\right]^{g(x)} =$ logarithmic differentiation (as it is complicated)

Steps to solve complicated derivatives:

1. ***ln*** *the both sides of the equation and use the laws of logarithms to simplify.*
2. *Differentiate implicitly with respect to* x *.*
3. *Solve the resulting equation for* y'*.*

If $f(x) < 0$ *for some values of* x *, then* $\ln(x)$ *is not defined, but we can write* $|\,y\,| = |\,f(x)\,|$.

Examples

Find the first derivative in each of the following :

$(1)\ y = x^{2x}$

Answer

Note: Ln both sides when the **base and power are variables**

$\therefore\ \ln y = \ln x^{2x} \Rightarrow \therefore\ \ln y = 2x \ln x$

Then differentiate w.r.to x : $\therefore\ \dfrac{y'}{y} = 2x\left[\dfrac{1}{x}\right] + \ln x\,[2] \Rightarrow \dfrac{y'}{y} = 2 + 2\ln x$

$\therefore\ y' = 2y(1+\ln x) \Rightarrow \boxed{\therefore\ y' = 2x^{2x}(1+\ln x)}$

$(2)\ y = (x)^{\frac{1}{x}}$

Answer

Note: Ln both sides when the **base and power are variables**

$\therefore\ \ln y = \ln (x)^{\frac{1}{x}} \Rightarrow \therefore\ \ln y = \dfrac{\ln x}{x}$

Then differentiate w.r.to x : $\therefore\ \dfrac{y'}{y} = \dfrac{x\left[\dfrac{1}{x}\right] - \ln x}{x^2} \Rightarrow \dfrac{y'}{y} = \dfrac{1-\ln x}{x^2}$

$\therefore\ y' = y\left[\dfrac{1-\ln x}{x^2}\right] \Rightarrow \boxed{\therefore\ y' = (x)^{\frac{1}{x}}\left[\dfrac{1-\ln x}{x^2}\right] = \sqrt[x]{x}\left[\dfrac{1-\ln x}{x^2}\right]}$

$(3)\ y = \left(x^3+5\right)^x$

Answer

Note: Ln both sides when the **base and power are variables**

$\therefore\ \ln y = \ln\left(x^3+5\right)^x \Rightarrow \therefore\ \ln y = x\ln\left(x^3+5\right)$

Then differentiate w.r.to x : $\therefore\ \dfrac{y'}{y} = x\left[\dfrac{3x^2}{x^3+5}\right] + \ln\left(x^3+5\right) \Rightarrow \therefore\ \dfrac{y'}{y} = \dfrac{3x^3}{x^3+5} + \ln\left(x^3+5\right)$

$\therefore\ y' = y\left[\dfrac{3x^3}{x^3+5} + \ln\left(x^3+5\right)\right] \Rightarrow \boxed{\therefore\ y' = \left(x^3+5\right)^x\left[\dfrac{3x^3}{x^3+5} + \ln\left(x^3+5\right)\right]}$

$(4)\ y=3(x)^{x^2+3}$

Answer

Note: Ln both sides when the **base and power are variables**

$\therefore\ \ln y=\ln 3(x)^{x^2+3} \Rightarrow \therefore\ \ln y=\ln 3+\ln(x)^{x^2+3} \Rightarrow \therefore\ \ln y=\ln 3+\left(x^2+3\right)\ln(x)$

Then differentiate w.r.to x : $\therefore\ \dfrac{y'}{y}=\left(x^2+3\right)\left[\dfrac{1}{x}\right]+\ln(x)\left[2x\right] \Rightarrow \therefore\ \dfrac{y'}{y}=\dfrac{x^2+3}{x}+2x\ln(x)$

$\therefore\ y'=y\left[\dfrac{x^2+3}{x}+2x\ln(x)\right] \Rightarrow \boxed{\therefore\ y'=3(x)^{x^2+3}\left[\dfrac{x^2+3}{x}+2x\ln(x)\right]}$

$(5)\ y=(\sin x)^{\tan x}$

Answer

Note: Ln both sides when the **base and power are variables**

$\therefore\ \ln y=\ln(\sin x)^{\tan x} \Rightarrow \therefore\ \ln y=\tan x\ \ln(\sin x)$

Then differentiate w.r.to x : $\therefore\ \dfrac{y'}{y}=\tan x\left[\dfrac{\cos x}{\sin x}\right]+\ln(\sin x)\left[\sec^2 x\right]$

$\therefore\ \dfrac{y'}{y}=1+\sec^2 x\ \ln(\sin x) \Rightarrow \therefore\ y'=y\left[1+\sec^2 x\ \ln(\sin x)\right]$

$\boxed{\therefore\ y'=(\sin x)^{\tan x}\left[1+\sec^2 x\ \ln(\sin x)\right]}$

$(6)\ y=x^{\cot x^2}$

Answer

Note: Ln both sides when the **base and power are variables**

$\therefore\ \ln y=\ln(x)^{\cot x^2} \Rightarrow \therefore\ \ln y=\cot\left(x^2\right)\ln(x)$

Then differentiate w.r.to x : $\therefore\ \dfrac{y'}{y}=\cot\left(x^2\right)\left[\dfrac{1}{x}\right]+\ln(x)\left[-2x\operatorname{cosec}^2\left(x^2\right)\right]$

$\therefore\ \dfrac{y'}{y}=\dfrac{\cot\left(x^2\right)}{x}-2x\operatorname{cosec}^2\left(x^2\right)\ln(x) \Rightarrow \therefore\ y'=y\left[\dfrac{\cot\left(x^2\right)}{x}-2x\operatorname{cosec}^2\left(x^2\right)\ln(x)\right]$

$\boxed{\therefore\ y'=(x)^{\cot x^2}\left[\dfrac{\cot\left(x^2\right)}{x}-2x\operatorname{cosec}^2\left(x^2\right)\ln(x)\right]}$

$(7)\ y^2 = 3^x \times 2^y$

Answer

1st method

Differentiate w.r.to x : $2y\dfrac{dy}{dx} = 3^x\left[2^y \ln 2 \dfrac{dy}{dx}\right] + 2^y\left[3^x \ln 3\right]$

$\therefore\ 2y\dfrac{dy}{dx} = 2^y 3^x \ln 2 \dfrac{dy}{dx} + 2^y 3^x \ln 3 \Rightarrow \therefore\ 2y\dfrac{dy}{dx} = y^2 \ln 2 \dfrac{dy}{dx} + y^2 \ln 3 \quad (\text{divide by } y)$

$\therefore\ 2\dfrac{dy}{dx} = y \ln 2 \dfrac{dy}{dx} + y \ln 3 \Rightarrow \therefore\ 2\dfrac{dy}{dx} - y \ln 2 \dfrac{dy}{dx} = y \ln 3$

$\therefore\ \dfrac{dy}{dx}\left[2 - y \ln 2\right] = y \ln 3 \Rightarrow \boxed{\therefore\ \dfrac{dy}{dx} = \dfrac{y \ln 3}{2 - y \ln 2}}$

2nd method [ln both sides]

$\therefore\ \ln y^2 = \ln 3^x \times 2^y \Rightarrow \therefore\ 2 \ln y = \ln 3^x + \ln 2^y \Rightarrow \therefore\ 2 \ln y = x \ln 3 + y \ln 2$

Then differentiate w.r.to x : $\therefore\ 2\dfrac{y'}{y} = \ln 3 + y' \ln 2 \Rightarrow \therefore\ 2\dfrac{y'}{y} = \ln 3 + y' \ln 2$

$\therefore\ 2y' = y \ln 3 + y y' \ln 2 \Rightarrow \therefore\ 2y' - y y' \ln 2 = y \ln 3$

$\therefore\ y'\left[2 - y \ln 2\right] = y \ln 3 \Rightarrow \boxed{\therefore\ y' = \dfrac{y \ln 3}{2 - y \ln 2}}$

$(8)\ y = (1-3x)^{\cos x}$

Answer

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.... $\boxed{\text{Ans : } y' = -(1-3x)^{\cos x}\left[\dfrac{3\cos(x)}{1-3x} + \sin(x)\ln(1-3x)\right]}$

$(9)\ y=(\ln x)^{\cos x}$

Answer

Note: Ln both sides when the **base and power are variables**

$\therefore \ln y=\ln(\ln x)^{\cos x} \Rightarrow \therefore \ln y=\cos x\ln(\ln x)$

Then differentiate w.r.to x : $\therefore \frac{y'}{y}=\cos x\left[\frac{\frac{1}{x}}{\ln x}\right]+\ln(\ln x)[-\sin x]$

$\therefore \frac{y'}{y}=\frac{\cos x}{x\ln x}-\sin x\ln(\ln x) \Rightarrow \therefore y'=y\left[\frac{\cos x}{x\ln x}-\sin x\ln(\ln x)\right]$

$$\therefore y'=(\ln x)^{\cos x}\left[\frac{\cos x}{x\ln x}-\sin x\ln(\ln x)\right]$$

$(10)\ y=\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}$

Answer

Note: The function is complicated product and quotient , so ln both sides to simplify

$\therefore \ln y=\ln\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5} \Rightarrow \therefore \ln y=\ln x^{\frac{3}{4}}\sqrt{x^2+1}-\ln(3x+2)^5$

$\therefore \ln y=\ln x^{\frac{3}{4}}+\ln(x^2+1)^{\frac{1}{2}}-\ln(3x+2)^5 \Rightarrow \therefore \ln y=\frac{3}{4}\ln x+\frac{1}{2}\ln(x^2+1)-5\ln(3x+2)$

Then differentiate w.r.to x : $\therefore \frac{y'}{y}=\frac{3}{4}\left[\frac{1}{x}\right]+\frac{1}{2}\left[\frac{2x}{x^2+1}\right]-\frac{5\times 3}{3x+2}$

$\therefore \frac{y'}{y}=\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2} \Rightarrow \therefore y'=y\left[\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right]$

$$\therefore y'=\frac{x^{\frac{3}{4}}\sqrt{x^2+1}}{(3x+2)^5}\left[\frac{3}{4x}+\frac{x}{x^2+1}-\frac{15}{3x+2}\right]$$
