Contents

Derivatives of Algebraic real function

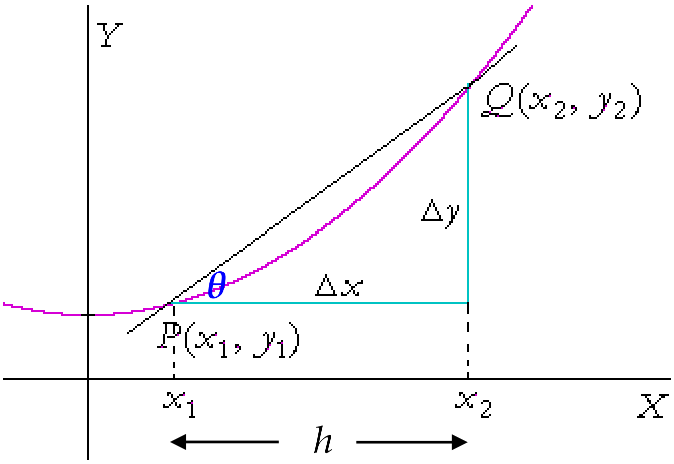
$$Rate\ of\ change = derivative\ of\ a\ function = f'(x) = \frac{dy}{dx}$$

A rate of change is a **rate** that describes how one quantity changes in relation to another quantity

Calculus however is concerned with rates of change that are not constant.

A secant is a straight line that cuts a curve.

(A tangent is a straight line that just touches a curve) , hence, consider a secant line that cuts the curve at points P and Q. The slope of the secant is the average rate of change between those two points $\frac{\Delta y}{\Delta x}$.



$$Slope\ of\ line = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\Delta y}{\Delta x} = \frac{f(x+h) - f(x)}{h}$$
$$= \frac{Opposite}{hypotonuse} = Tan\theta$$

let the point Q approach the point P, and let us calculate the average rate of change as x_2 gets closer and closer and closer to x_1. That is, let x_2 approach x_1 ,In other words, let Δx approach 0. " $h \to 0$ "

$$\textbf{Slope of any function} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f'(x) = \frac{dy}{dx}$$

Solved examples

Example (1)

Let $f(x) = 2x^2 + 5x - 1$, then find the rate of change of f.

Answer

$\because f(x) = 2x^2 + 5x - 1$ and $\because$ the rate of change of $f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\therefore f(x+h) = 2(x+h)^2 + 5(x+h) - 1 = 2\left(x^2 + 2xh + h^2\right) + 5x + 5h - 1$$

$$= 2x^2 + 4xh + 2h^2 + 5x + 5h - 1$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 + 5x + 5h - 1 - 2x^2 - 5x + 1}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{4xh + 2h^2 + 5h}{h} = \lim_{h \to 0} \frac{h(4x + 2h + 5)}{h} = \lim_{h \to 0} 4x + 2h + 5$$

$$\boxed{\therefore f'(x) = 4x + 5}$$

Example (2)

Let $f(x) = x^3$, then find the rate of change of f.

Answer

$\because f(x) = x^3$ and $\because$ the rate of change of $f(x) = \lim\limits_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\therefore f(x+h) = (x+h)^3 = (x+h)^2(x+h) = \left(x^2 + 2xh + h^2\right)(x+h) = x^3 + 3x^2h + 3xh^2 + h^3$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x^3}{h}$$

$$\therefore f'(x) = \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3}{h} = \lim_{h \to 0} \frac{h\left(3x^2 + 3xh + h^2\right)}{h} = \lim_{h \to 0} 3x^2 + 3xh + h^2$$

$$\boxed{\therefore f'(x) = 3x^2}$$

So, from now on, we don't need to use the rule of rate of change to get the differentiation of any function (as it is a long method).
All we need is to use the following simple methods:

There are various shapes of derivatives:

$$1^{st}\text{ derivative} = \text{rate of change} = f' = y' = \frac{dy}{dx} = \frac{d\ f(x)}{dx} = \text{Slope of the tangent}$$

Rules of derivatives:

(1) If $f(x) = x^n \Rightarrow f'(x) = \frac{dy}{dx} = n\ x^{n-1}$

(2) If $y = a\ x^n$, where a is constant $\Rightarrow y' = \frac{dy}{dx} = a\,n\ x^{n-1}$

(3) If $y = \text{constant} \Rightarrow y' = \frac{dy}{dx} = \text{Zeroooooo}$

(4) Special case If $y = \sqrt{g(x)} \Rightarrow y' = \frac{dy}{dx} = \frac{g'(x)}{2\sqrt{g(x)}}$

Example (1)

find the derivative of each of the following function

(1) $y = \frac{3}{4}x^{-2}$

Answer

$$\frac{dy}{dx} = -2 \times \frac{3}{4}(x)^{-2-1} = -2 \times \frac{3}{4}(x)^{-3} = \frac{-3}{2}x^{-3}$$

(2) $y = 2\,x^5 + 3\,x^3 - 4\,x + 6$

Answer

$$\frac{dy}{dx} = 10\ x^4 + 9\ x^2 - 4$$

(3) $f(x) = \sqrt{x}$

Answer

$$\because\ f(x) = \sqrt{x} = x^{1/2} \Rightarrow \therefore\ f'(x) = \frac{1}{2} \times (x)^{\frac{1}{2}-1} = \frac{1}{2} \times (x)^{-1/2} = \frac{1}{2\sqrt{x}}$$

(4) $f(x) = 6\sqrt{2x+1}$

Answer

$$f'(x) = 6 \times \frac{2}{2\sqrt{2x+1}} = \frac{6}{\sqrt{2x+1}}$$

(5) $f(x) = 3\sqrt[3]{x} + \frac{1}{2\sqrt[3]{x^2}}$

Answer

$$f(x) = 3x^{\frac{1}{3}} + \frac{1}{2}x^{-2/3} \Rightarrow f'(x) = 3 \times \frac{1}{3}x^{\frac{1}{3}-1} + \frac{-2}{3} \times \frac{1}{2}(x)^{\frac{-2}{3}-1} = x^{\frac{-2}{3}} + \frac{-1}{3}(x)^{-5/3} = \frac{1}{\sqrt[3]{x^2}} - \frac{1}{3\sqrt[3]{x^5}}$$

$(6)\ f(x) = 2x\sqrt{x} + 4\sqrt{x} - 1$

Answer

$$\because\ f(x) = 2\,x\sqrt{x} + 4\sqrt{x} - 1 \Rightarrow\ 2x^{3/2} + 4\sqrt{x} - 1$$

$$f'(x)\ = \frac{3}{2} \times 2\,x^{1/2} + 4 \times \frac{1}{2\sqrt{x}} = 3\sqrt{x} + \frac{2}{\sqrt{x}}$$

$(7)\ g(x) = x - \frac{2}{x} + \frac{3}{x^2} + \frac{x}{5}$

Answer

$$\because\ g(x) = x - 2x^{-1} + 3x^{-2} + \frac{1}{5}x\ \Rightarrow\ \therefore\ g'(x) = 1 + 2x^{-2} - 6x^{-3} + \frac{1}{5} = 1 + \frac{2}{x^2} - \frac{6}{x^3} + \frac{1}{5}$$

$(8)\ g(x) = \frac{2}{x^2}\left(x^4 - 4x^2 + x - 3\right)$

Answer

$$\because\ g(x) = 2x^2 - 8 + \frac{2}{x} - \frac{6}{x^2} = 2x^2 - 8 + 2x^{-1} - 6x^{-2}\ \Rightarrow\ g'(x) = 4x - 2x^{-2} + 12x^{-3} = 4x - \frac{2}{x^2} + \frac{12}{x^3}$$

$(9)\ g(x) = \left(4x^2 + 1\right)\left(1 - 4x^2\right) + 2$

Answer

$$\because\ g(x) = 4x^3 - 8x^4 + x - 2x^2 + 2\ \Rightarrow\ g'(x) = 12x^2 - 32x^3 + 1 - 4x$$

$(10)\ g(x) = \frac{5x^5 + 10x^4 - 2x^3 + 10x^2 - 20}{10x^3}$

Answer

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The derivative of two product or division functions

Rules

(1) If $y=(ax+b)^n \Rightarrow$ $\therefore \frac{dy}{dx}=n(ax+b)^{n-1}(a)$

(2) The first derivative of the product of the two functions :

So If $f=x\times y \Rightarrow$ $f'=xy'+yx'$

(3) The first derivative of the quotient "Divisions" of the two functions :

So If $f=\frac{x}{y} \Rightarrow$ $f'=\frac{yx'-xy'}{y^2}$

Solved examples

Example (1)

Find the first derivative of: $y=(2x+5)^{12}$

Answer

$$\frac{dy}{dx}=12(2x+5)^{11}\times 2=24(2x+5)^{11}$$

Example (2)

Find the first derivative of : $y=(x^3+5x^2-4)^6$

Answer

$$\frac{dy}{dx}=6(x^3+5x^2-4)^5\times(3x^2+10x)=6\times(3x^2+10x)(x^3+5x-4)^5$$

Example (3)

Find the first derivative of: $y=(1-x)^3(1+x)^3$

Answer

$$y=(1-x)^3(1+x)^3=((1-x)(1+x))^3=(1-x^2)^3=3(1-x^2)^2(2x)=6x(1-x^2)^2$$

Example (4)

Find the first derivative of: $y=(x+5)(3x-1)^5$

Answer

$$\frac{dy}{dx}=(x+5)\left[5(3x-1)^4\times 3\right]+(3x-1)^5(1)=15(x+5)(3x-1)^4+(3x-1)^5$$

Example (5)

Find the first derivative of: $y = \dfrac{8x}{\sqrt[3]{x^2 - 2}}$

Answer

$$\because\ y = \frac{8x}{\left(x^2-2\right)^{\frac{1}{3}}} \Rightarrow \frac{dy}{dx} = \frac{8\left(\sqrt[3]{x^2-2}\right) - 8x\left(\frac{1}{3}\left(x^2-2\right)^{\frac{-2}{3}}(2x)\right)}{\sqrt[3]{\left(x^2-2\right)^2}} = \frac{8\sqrt[3]{x^2-2} - \frac{16}{3}x^2\left(x^2-2\right)^{\frac{-2}{3}}}{\sqrt[3]{\left(x^2-2\right)^2}}$$

Example (6)

Find the first derivative of: $y = \left(\dfrac{2x+1}{3x-1}\right)^4$

Answer

$$\because\ \frac{dy}{dx} = 4\left(\frac{2x+1}{3x-1}\right)^3 \times \frac{2(3x-1) - 3(2x+1)}{(3x-1)^2} \Rightarrow \therefore\ \frac{dy}{dx} = 4\left(\frac{2x+1}{3x-1}\right)^3 \times \frac{-5}{(3x-1)^2}$$

Example (7)

Find the first derivative of: $y = \dfrac{(2x+1)^2}{(3x+1)^3}$

Answer

$$\because\ \frac{dy}{dx} = \frac{(3x+1)^3\, 2(2x+1)(2) - (2x+1)^2 (3)(3x+1)^2 (3)}{(3x+1)^6}$$

$$= \frac{4(3x+1)^3 (2x+1) - 9(2x+1)^2 (3x+1)^2}{(3x+1)^6}$$

Example (8)

Find the first derivative of: $y = \sqrt{\left(x^2 - 4x + 7\right)^3}$

Answer

$$\because\ y = \left(x^2-4x+7\right)^{\frac{3}{2}} \Rightarrow \therefore\ \frac{dy}{dx} = \frac{3}{2}\left(x^2-4x+7\right)^{\frac{3}{2}-1}(2x-4) = \frac{3}{2}\left(x^2-4x+7\right)^{\frac{1}{2}}(2)(x-2)$$

$$\therefore\ \frac{dy}{dx} = 3\left(x^2-4x+7\right)^{\frac{1}{2}}(x-2) = 3(x-2)\sqrt{x^2-4x+7}$$

Example (9)

Find the first derivative of: $y = x^3 \sqrt[5]{\left(x^2+2x+2\right)^2}$

Answer

$$\because\ y = x^3 \sqrt[5]{\left(x^2+2x+2\right)^2} = x^3\left(x^2+2x+2\right)^{\frac{2}{5}}$$

$$\therefore\ \frac{dy}{dx} = x^3 \times \frac{2}{5}\left(x^2+2x+2\right)^{\frac{-3}{5}}\left(2x+2\right) + \left(x^2+2x+2\right)^{\frac{2}{5}} \times 3x^2$$

$$= \frac{2x^3\left(2x+2\right)}{5\sqrt[5]{\left(x^2+2x+2\right)^3}} + 3x^2\sqrt[5]{\left(x^2+2x+2\right)^2}$$

Example (10)

If $y = \sqrt{\frac{1-x}{1+x}}$, then find $\frac{dy}{dx}$

Answer

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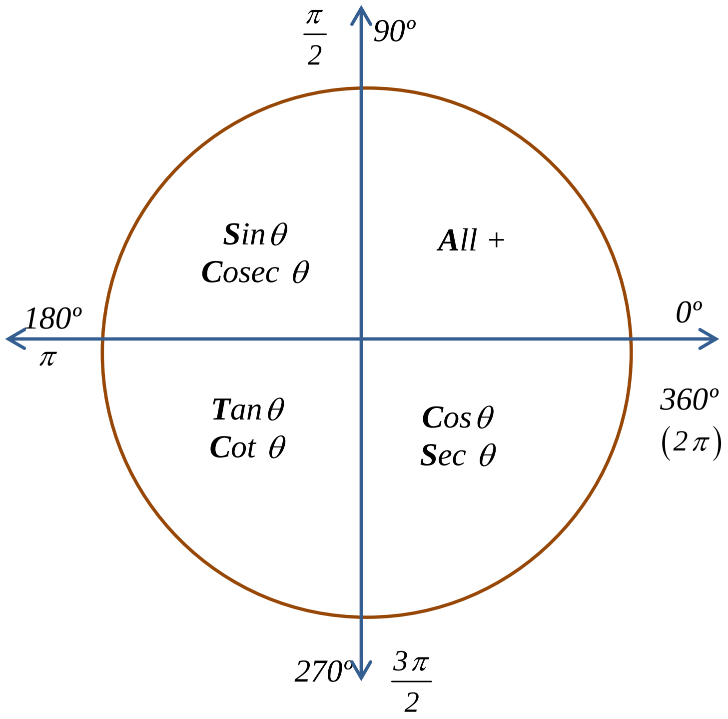
The derivative of trigonometric functions

But before this, we need to revise some trigonometric relations and rules

Remember that



Relation between the basic trigonometric functions

$Tan\ \theta = \frac{Sin\theta}{Cos\theta}$ | $Cot\theta = \frac{Cos\theta}{Sin\theta}$ | $Cos\theta = \frac{1}{Sec\theta}$ | $Sin\theta = \frac{1}{Cosec\ \theta}$ | $Cot\theta = \frac{1}{Tan\theta}$

$Tan\ \theta = \frac{1}{Cot\theta}$ | $Sec\theta = \frac{1}{Cos\theta}$ | $Cosec\ \theta = \frac{1}{Sin\theta}$

$Cos\ (180^o - \theta) = -Cos\theta$	$Sec\ (270^o + \theta) = Cosec\theta$	$Sin\ (270^o - \theta) = -Cos\theta$
$Sin\ (180^o + \theta) = -Sin\ \theta$	$Cos(360^o - \theta) = Cos\theta$	$Tan\ (90^o - \theta) = Cot\ \theta$
$Cosec\ (90^o + \theta) = Sec\theta$	$Cos(-\theta) = Cos\theta$	$Sin\ (-\theta) = -Sin\ \theta$

Important rules

$\because \ Sin^2\theta + Cos^2\theta = 1$

$\therefore \ Cos^2\theta = 1 - Sin^2\theta$

$Or \ \ Sin^2\theta = 1 - Cos^2\theta$

$1 + Tan^2\theta = Sec^2\theta$

$1 + Cot^2\theta = Cosec^2\theta$

Remarks : $Cos^2\theta = (Cos\theta)^2$ $\quad Sin^2\theta = (Sin\theta)^2$ $\quad Tan^2\theta = (Tan\theta)^2$

The basic trigonometric angles are 30º , 45º , 60º , 90º , 180º , 270º , 360º

Trigonometric function of sum & different between two angles

Rules " Study well "

(1) $Sin(A+B) = Sin\,A\,Cos\,B + Cos\,A\,Sin\,B$

(2) $Sin(A-B) = Sin\,A\,Cos\,B - Cos\,A\,Sin\,B$

(3) $Cos(A+B) = Cos\,A\,Cos\,B - Sin\,A\,Sin\,B$

(4) $Cos(A-B) = Cos\,A\,Cos\,B + Sin\,A\,Sin\,B$

(5) $Tan\,(A+B) = \dfrac{Tan\,A + Tan\,B}{1 - Tan\,A\,Tan\,B}$

(6) $Tan\,(A-B) = \dfrac{Tan\,A - Tan\,B}{1 + Tan\,A\,Tan\,B}$

(7) $Cot\,(A+B) = \dfrac{1 - Tan\,A\,Tan\,B}{Tan\,A + Tan\,B}$

(8) $Cot\,(A-B) = \dfrac{1 + Tan\,A\,Tan\,B}{Tan\,A - Tan\,B}$

Trigonometric function of Double the angle

1 $Sin\,2A = 2\,Sin\,A\,Cos\,A$

Note $Sin\,A = 2\,Sin\dfrac{A}{2}Cos\dfrac{A}{2}$

2 $Cos\,2A = Cos^2A - Sin^2A$

$= 2Cos^2A - 1 = 1 - 2\,Sin^2A$

Note $Cos\,A = Cos^2\dfrac{A}{2} - Sin^2\dfrac{A}{2}$

$= 1 - 2\,Sin^2\dfrac{A}{2} = 2Cos^2\dfrac{A}{2} - 1$

3 $Tan\,2A = \dfrac{2\,Tan\,A}{1 - Tan^2A}$

The derivative of trigonometric functions

Rules

(1) If $y = Sin\ x \Rightarrow \dfrac{dy}{dx} = Cos\ x$

(2) If $y = Cos\ x \Rightarrow \dfrac{dy}{dx} = -Sin\ x$

(3) If $y = Tan\ x \Rightarrow \dfrac{dy}{dx} = Sec^2 x$

(4) If $y = Sin(ax+b) \Rightarrow y' = a\,Cos\,(ax+b)$

(5) If $y = Cos(ax+b) \Rightarrow y' = -a\,Sin\,(ax+b)$

(6) If $y = Tan(ax+b) \Rightarrow y' = a\,Sec^2\,(ax+b)$

Solved examples

Example (1)

Find $\dfrac{d}{dx}\ Sec\ x$

Answer

$$\because\ f(x) = Sec\,x = \frac{1}{Cos\,x} \Rightarrow f'(x) = \frac{Cos\,x\,(0) - (1)(-Sin\,x)}{(Cos\,x)^2}$$

$$= \frac{Sin\,x}{(Cos\,x)^2} = \frac{Sin\,x}{Cos\,x\ Cos\,x} = Tan\ x\ Secx$$

Example (2)

Find $\dfrac{d}{dx}\ Cosec\ x$

Answer

$$\because\ f(x) = Cosec\,x = \frac{1}{Sin\,x} \Rightarrow \therefore\ f'(x) = \frac{(Sin\,x)(0) - (Cos\,x)}{(Sin\,x)^2} = \frac{-Cos\,x}{(Sin\,x)^2} = \frac{-Cos\,x}{Sin\,x\ Sin\,x}$$

$$= -\,Cot\ x\ Cosec\,x$$

Example (3)

Find the derivative of : $y = x\ Sin\ 2x + Cos\dfrac{\pi}{4}$

Answer

$$\frac{dy}{dx} = x(2Cos\ 2x) + (1)Sin\ 2x = 2\ Cos\ 2x\ + Sin\ 2x$$

Example (4)

Find the derivative of : $y=\frac{Cos\ x}{x}+Tan3x$

Answer

$$\frac{dy}{dx}=\frac{\left(x(-Sin\,x)-Cos\ x\right)}{x^2}+3Sec^2 3x=\frac{\left(-x\ Sin\,x-Cos\ x\right)}{x^2}+3Sec^2 3x$$

Example (5)

Find the derivative of : (a) $f(x)=\left(Sin\,x^2\right)$ (b) $f(x)=Sin^2 2x$

Answer (a)

$$f'(x)=2x\ Cos\ x^2$$

Answer (b)

$$\because\ f(x)=\left(Sin^2 2x\right)=\left(Sin\,2x\right)^2$$

$$\therefore\ f'(x)=2\left(Sin\,2x\right)(2)Cos\,2x$$

Example (6)

If $y=x+x\ Sin^2 2x$, then **prove that :** $\frac{dy}{dx}=1+2x\,Sin4x+Sin^2 2x$

Answer

$$\because\ f(x)=x+x\,Sin^2 2x=x+x\left(Sin\,2x\right)^2$$

$$\therefore\ f'(x)=1+x\left[2\left(Sin2x\right)2Cos\,2x\right]+Sin^2 2x=1+x\left[4\,Sin\,2x\,Cos\,2x\right]+Sin^2 2x$$

$$\therefore\ f'(x)=1+2x\,Sin\,4x+Sin^2 2x$$

Example (7)

If $y=\left(Sin\theta+Cos\theta\right)^2$, **prove that :** $\frac{dy}{dx}=2Cos\,2\theta$

Answer

$$\therefore\frac{dy}{dx}=2\left(Sin\theta+Cos\theta\right)\left(Cos\theta-Sin\theta\right)=2\left(Sin\theta Cos\theta-Sin^2\theta+Cos^2\theta\ -Sin\theta Cos\theta\right)$$

$$=2\left(-Sin^2\theta\ +Cos^2\theta\right)=2\left(Cos^2\theta-Sin^2\theta\right)=2Cos\,2\theta$$

Example (8)

Find the first derivative of : $y=12\ x\ Sin\ x\ Cos\ x+Tan\sqrt{x}$

Answer

$$\because\ y=12\ x\ Sin\ x\ Cos\ x=6\,x\left(2\,Sin\ x\ Cos\ x\right)+Tan\sqrt{x}\ \Rightarrow\ \therefore\ y=6x\ Sin\ 2x+Tan\sqrt{x}$$

$$\therefore\frac{dy}{dx}=6\ x\left(2\ Cos\ 2x\right)+6\left(Sin\,2x\right)+\frac{1}{2\sqrt{x}}Sec^2\sqrt{x}$$

$$\therefore\frac{dy}{dx}=12\ x\ Cos\ 2x+6\ Sin\ 2x+\frac{1}{2\sqrt{x}}Sec^2\sqrt{x}$$

Example (9)

If $y = 2\,Cos\,x\,Sin\,\frac{x}{2}\,Cos\frac{x}{2}$, then **prove that :** $\frac{dy}{dx} = Cos\,2x$

Answer

$$\because\ y = \boxed{2}\,Cos\,x\,\boxed{Sin\frac{x}{2}Cos\frac{x}{2}} = Cos\,x\,Sin\,x$$

$$\therefore\ \frac{dy}{dx} = Cos\,x\left(Cos\,x\right) + Sin\,x\left(-Sin\,x\right) = Cos^2 x - Sin^2 x = Cos\,2x$$

Example (10)

Find the derivative of : $y = x\sqrt{1 + Cos\,x}$

Answer

$$\because\ Cos\,x = 2\,Cos^2\frac{1}{2}x - 1 \Rightarrow \therefore\ y = x\sqrt{1 + 2\,Cos^2\frac{1}{2}x - 1} = x\sqrt{2\,Cos^2\frac{1}{2}x} = \sqrt{2}\,x\,Cos^2\frac{1}{2}x$$

$$\therefore\ \frac{dy}{dx} = \sqrt{2}\,x\left(2\,Cos\frac{1}{2}x\right)\left(-Sin\frac{1}{2}x\right) + \sqrt{2}\,Cos^2\frac{1}{2}x = -\sqrt{2}\,x\left[2\,Sin\frac{1}{2}x\,Cos\frac{1}{2}x\right] + \sqrt{2}\,Cos^2\frac{1}{2}x$$

$$\therefore\ \frac{dy}{dx} = -\sqrt{2}\,x\,Sin\,x + \sqrt{2}\,Cos^2\frac{1}{2}x$$

Example (11)

Find the derivative of : $y = \frac{Sin\,(2x+1)}{Cos(3x-1)}$

Answer

$$\frac{dy}{dx} = \frac{Cos(3x-1)\times 2\,Cos(2x+1) + 3\,Sin\,(2x+1)\,Sin(3x-1)}{Cos^2(3x-1)}$$

$$= \frac{2\,Cos(2x+1)\,Cos(3x-1) + 3\,Sin(2x+1)\,Sin(3x-1)}{Cos^2(3x-1)}$$

Example (12)

Find $\frac{dy}{dx}$ of the function $f(x) = \frac{1 + Cos\,x}{Sin\,x}$, then **prove that :**

$f(x) = -Cosec\,x\,(Cosec\,x + Cot\,x)$

Answer

$$\because\ f(x) = \frac{1 + Cos\,x}{Sin\,x} \rightarrow \therefore\ f(x) = \frac{Sin\,x(-Sin\,x) - (1 + Cos\,x)(Cos\,x)}{Sin^2 x}$$

$$\therefore\ f(x) = \frac{-Sin^2 x - Cos^2 x - Cos\,x}{Sin^2 x} = \frac{-(Sin^2 x + Cos^2 x) - Cos\,x}{Sin^2 x} = \frac{-1 - Cos\,x}{Sin^2 x}$$

$$\therefore\ f(x) = \frac{-1}{Sin^2 x} + \frac{-Cos\,x}{Sin^2 x} = -Cosec^2 x - Cosec\,x\,Cot\,x = -Cosec\,x\,(Cosec\,x + Cot\,x)$$

Example (13)

If $y=\left(\frac{1+\sin x}{1-\sin x}\right)^4$, then find $\frac{dy}{dx}$

Answer

$$\frac{dy}{dx}=4\left(\frac{1+Sin\,x}{1-Sin\,x}\right)^3\times\frac{(1-Sin\,x)\,Cos\,x-(1+Sin\,x)(-Cos\,x)}{(1-Sin\,x)^2}$$

$$\therefore 4\left(\frac{1+Sin\,x}{1-Sin\,x}\right)^3\times\frac{(Cos\,x-Sin\,x\,Cos\,x+Cos\,x-Sin\,x\,Cos\,x)}{(1-Sin\,x)^2}=\left(\frac{1+Sin\,x}{1-Sin\,x}\right)^3\times\frac{8Cos\,x}{(1-Sin\,x)^2}$$

Example (14)

If $y=(2x+1)^4\,Sin3x$, then find $\frac{dy}{dx}$

Answer

$$\therefore\frac{dy}{dx}=(2x+1)^4\times 3Cos3x+4(2x+1)^3\times 2\times Sin3x=3(2x+1)^4\,Cos\,3x+8(2x+1)^3\,sin3x$$

Chain rule (Composite function)

Rules

If $y=f(z)$ and $Z=f(x)$

$$\therefore\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}$$

Example (1)

let $y=Z^8+1$ and $Z=2x-3$, find $\frac{dy}{dx}$ at $(2,3)$

Answer

For $y=Z^8+1\Rightarrow\frac{dy}{dz}=8\,Z^7---(1)$

And for $Z=2x-3\quad\Rightarrow\frac{dz}{dx}=2---(2)$

From (1) and (2): we get $\therefore\frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=(8Z^7)\times 2=16Z^7$

And by substitution : $\frac{dy}{dx}=16(2x-3)^7\rightarrow$ $\boxed{\therefore\frac{dy}{dx}=16(2(2)-3)^7=16}$

Example (2)

Let $y = 3Z + \frac{2}{Z}$, $Z = \frac{1}{2x+3}$, $x \neq \frac{-3}{2}$, find $\frac{dy}{dx}$

Answer

$$\frac{dy}{dz} = 3 - \frac{2}{Z^2} = 3 - 2(2x+3)^2 = 3 - 2\left(4x^2 + 12x + 9\right)$$

$$\frac{dz}{dx} = -\frac{2}{(2x+3)^2} \Rightarrow \therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \left(-8x^2 - 24x - 15\right) \times \frac{-2}{(2x+3)^2}$$

$$= 2\left(8x^2 + 24x + 15\right)(2x+3)^{-2}$$

Example (3)

If $y = \frac{z-1}{z+1}$, $z = \frac{x+1}{x-1}$, $x \neq 1$, then find $\frac{dy}{dx}$

Answer

$$\frac{dy}{dz} = \frac{(z+1)-(z-1)}{(z+1)^2} = \frac{2}{(z+1)^2} = \frac{2}{\left[\left(\frac{x+1}{x-1}\right)+1\right]^2} = \frac{2(x-1)^2}{(x+1+x-1)^2} = \frac{(x-1)^2}{2x^2}$$

$$\therefore \frac{dz}{dx} = \frac{(x-1)-(x+1)}{(x-1)^2} = \frac{-2}{(x-1)^2} \Rightarrow \therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \frac{(x-1)^2}{2x^2} \times \frac{-2}{(x-1)^2} = \frac{-1}{x^2} = -x^{-2}$$

Example (4)

If $y = \sqrt{Z} + \frac{1}{\sqrt{Z}}$ and $Z = x^2 + 7$, find $\frac{dy}{dx}$ at $x = 3$

Answer

$$\because y = Z^{\frac{1}{2}} + Z^{-\frac{1}{2}} \rightarrow \therefore \frac{dy}{dz} = \frac{1}{2\sqrt{Z}} - \frac{1}{2Z\sqrt{Z}}$$

$$\because Z = x^2 + 7 \rightarrow \therefore \frac{dz}{dx} = 2x \Rightarrow \therefore \frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = 2x\left(\frac{1}{2\sqrt{Z}} - \frac{1}{2Z\sqrt{Z}}\right)$$

$$\therefore \frac{dy}{dx} = \frac{x}{\sqrt{Z}}\left(1 - \frac{1}{Z}\right)$$, so when $x = 3 \Rightarrow$ then $Z = 16 \rightarrow$ $\boxed{\therefore \frac{dy}{dx} = \frac{3}{4}\left(1 - \frac{1}{16}\right) = \frac{45}{64}}$

Example (5)

If $x=\sqrt{2Z-3}$ and $y=2Z^2-1$, **prove that :** $\frac{dy}{dx}=2x\left(x^2+3\right)$

Answer

$\because\ x=(2Z-3)^{\frac{1}{2}} \Rightarrow \therefore \frac{dx}{dz}=\frac{1}{2}(2Z-3)^{\frac{-1}{2}}(2)=(2Z-3)^{\frac{-1}{2}} \Rightarrow \boxed{\frac{dx}{dz}=\frac{1}{\sqrt{(2Z-3)}}}$

And $\because\ y=2Z^2-1 \Rightarrow \boxed{\therefore \frac{dy}{dz}=4z} \Rightarrow \boxed{\therefore \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=4Z\sqrt{(2Z-3)}---(1)}$

And $\because\ x=(2Z-3)^{\frac{1}{2}}$ "by squaring both sides" $\Rightarrow x^2=2Z-3 \Rightarrow 2Z=x^2+3$

$\boxed{\therefore Z=\frac{x^2+3}{2}---(2)} \Rightarrow$ then by substituting (2) in (1) :

$\therefore \frac{dy}{dx}=4\left[\frac{x^2+3}{2}\right]\sqrt{\left(2\left[\frac{x^2+3}{2}\right]-3\right)}=2\left(x^2+3\right)\sqrt{x^2+\cancel{3}-\cancel{3}}=2x\left(x^2+3\right)=R.H.S$

Example (6)

If $y=\sqrt{1+Z^2}$ and $Z=\frac{1}{x}$, then prove that : $\frac{dy}{dx}=\frac{-1}{x^2\sqrt{1+x^2}}$

Answer

$\frac{dy}{dz}=\frac{1}{2}\left(1+z^2\right)^{\frac{-1}{2}}\times 2z=\frac{z}{\sqrt{1+z^2}}$ And $\because\ Z=x^{-1} \Rightarrow \frac{dz}{dx}=\frac{-1}{x^2}$

$\therefore \frac{dy}{dx}=\frac{dy}{dz}\times\frac{dz}{dx}=\frac{z}{\sqrt{1+z^2}}\times\frac{-1}{x^2}=\frac{-z}{x^2\sqrt{1+z^2}}=\frac{-x^{-1}}{x^2\sqrt{1+x^{-2}}}$

$\therefore \frac{dy}{dx}=\frac{-1}{x^3\sqrt{1+\frac{1}{x^2}}}=\frac{-1}{x^2\times x\sqrt{1+\frac{1}{x^2}}}=\frac{-1}{x^2\sqrt{1+x^2}}$

Example (7)

If $y=\sqrt{2x^2+7}$ and $Z=(1-2x)^2$, prove that : $\frac{dz}{dx}+4\left(1-y\frac{dy}{dx}\right)=0$

Answer

$\because y=\left(2x^2+7\right)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx}=\frac{1}{2}\left(2x^2+7\right)^{-\frac{1}{2}}\times 4x=\frac{4x}{2\sqrt{2x^2+7}}=\frac{2x}{\sqrt{2x^2+7}}$

$\therefore \frac{dz}{dx}=2(1-2x)\times -2=-4(1-2x)$

$\therefore \frac{dz}{dx}+4\left(1-y\frac{dy}{dx}\right)=(-4+8x)+4-4\times\sqrt{2x^2+7}\times\frac{2x}{\sqrt{2x^2+7}}=8x-8x=0$

Summary of differentiation rules

1^{st} derivative = rate of change = $f' = y' = \frac{dy}{dx} = \frac{d}{dx} f(x)$ = slope of a line

Also , we have taken some rules of differentiation which are :

A function	Its derivative
(1) If $f(x) = x^n$	$f'(x) = n\,x^{n-1}$
(2) If $y = a\,x^n$, where a is constant	$y' = a\,n\,x^{n-1}$
(3) If $y = constant$	$\frac{dy}{dx} = Zerooooo$
(4) If $y = Sin\ x$	$y' = Cos\ x$
(5) If $y = Cos\ x$	$y' = -Sin\ x$
(6) If $y = Tan\ x$	$y' = Sec^2\ x$
(7) If $y = Sin\ (a\,x + b)$	$y' = a\ Cos\ (a\,x + b)$
(8) If $y = Cos\ (a\,x + b)$	$y' = -a\ Sin\ (a\,x + b)$
(9) If $y = Tan\ (a\,x + b)$	$y' = a\ Sec^2\ (a\,x + b)$
(10) If $f = (a\,x + b)^n$	$f' = a\,n\,(a\,x + b)^{n-1}$
(11) If $f = \sqrt{g(x)}$	$f' = \frac{g'(x)}{2\sqrt{g(x)}}$
(12) If $f = x \times y$	$f' = x\,y' + y\,x'$
(13) $f = \frac{x}{y}$	$f' = \frac{y\,x' - x\,y'}{y^2}$

This year, we will discuss three new kinds of differentiation

Derivative of the reciprocal trigonometric functions

Derivative of the Exponential functions

Derivative of the Logarithmic functions

1 LESSON

The derivative of the reciprocal of trigonometric functions

Rules

A function	Its derivative
(1) If $f(x) = Sec\,x$	$f'(x) = Tan\,x\,Sec\,x$
(2) If $f(x) = Cosec\,x$	$f'(x) = -\,Cosec\,x\,Cot\,x$
(3) If $f(x) = Cot\,x$	$f'(x) = -\,Cosec^2\,x$
(4) If $f(x) = Sec\,a\,x$	$f'(x) = a\,Tan\,a\,x\,Sec\,a\,x$
(5) If $f(x) = Cosec\,a\,x$	$f'(x) = -a\,Cosec\,a\,x\,Cot\,a\,x$
(6) If $f(x) = Cot\,a\,x$	$f'(x) = -a\,Cosec^2\,a\,x$

Examples

Find the first derivative of each of the following functions:

(1) $y = Sec(5x^2 + 2)$

Answer

$$\frac{dy}{dx} = 10x\,Sec(5x^2+2)\,Tan(5x^2+2)$$

(2) $y = 3x^5 + 2Sin\,3x + 4Cot\,x$

Answer

$$\frac{dy}{dx} = 15x^4 + 2Cos\,3x\,[3] - 4Cosec^2 x$$

$$\frac{dy}{dx} = 15x^4 + 6\,Cos\,3x - 4Cosec^2 x$$

(3) $y = 3\,Sec\,2x + 2\,Cosec\,3x$

Answer

$$\frac{dy}{dx} = 3[Sec\,2x\,Tan\,2x]\times 2 - 2[Cosec\,3x\,Cot\,3x]\times 3$$

$$\frac{dy}{dx} = 6\,Sec\,2x\,Tan\,2x - 6\,Cosec\,3x\,Cot\,3x$$

(4) $y = Cot(x^2 + 2x)$

Answer

....

....

(5) $y = Sec\sqrt{x^2-2} + Cos7x$

Answer

$$\frac{dy}{dx} = \left[Sec\sqrt{x^2-2}\;Tan\sqrt{x^2-2}\right]\times\frac{2x}{2\sqrt{x^2-2}} - 7\,Sin7x$$

$$\frac{dy}{dx} = \frac{x\,Sec\sqrt{x^2-2}\;Tan\sqrt{x^2-2}}{\sqrt{x^2-2}} - 7\,Sin7x$$

(6) $y = x\,Cosec\,x$

Answer

$$\frac{dy}{dx} = x[-Cosec\,x\,Cot\,x] + Cosec\,x\,[1]$$

$$\frac{dy}{dx} = -x\,Cosec\,x\,Cot\,x + Cosec\,x$$

$(7)\ y = Cosec\,3x\,Cot\left(\pi - \frac{1}{x}\right)$

Answer

$$\frac{dy}{dx} = Cosec\,3x\left[-Cosec^2\left(\pi - \frac{1}{x}\right)\times\left(\frac{1}{x^2}\right)\right] + Cot\left(\pi - \frac{1}{x}\right)[-3Cosec\,3x\,Cot\,3x]$$

$$\frac{dy}{dx} = \frac{-Cosec\,3x\,Cosec^2\left(\pi - \frac{1}{x}\right)}{x^2} - 3Cosec\,3x\,Cot\,3x\,Cot\left(\pi - \frac{1}{x}\right)$$

$(8)\ y = Cot\,(Cos\,3x)$

Answer

$$\frac{dy}{dx} = -Cosec^2(Cos\,3x)\times(-3Sin\,3x)$$

$$\frac{dy}{dx} = 3\,Sin\,3x\,Cosec^2(Cos\,3x)$$

$(9)\ y = Sin\,(Cosec\,3x^2)$

Answer

$$\frac{dy}{dx} = Cos(Cosec\,3x^2)\times(-Cosec\,3x^2\,Cot\,3x^2)[6x]$$

$$\frac{dy}{dx} = -6x\,Cosec\,3x^2\,Cot\,3x^2\ Cos(Cosec\,3x^2)$$

$(10)\ y = \frac{1}{(Cosec\,2x + Cot\,3x)}$

Answer

$$Let\ y = (Cosec\,2x + Cot\,3x)^{-1}$$

$$\frac{dy}{dx} = -(Cosec\,2x + Cot\,3x)^{-2}\left[-2Cosec\,2x\,Cot\,2x - 3Cosec^2 3x\right] = \frac{2Cosec\,2x\,Cot\,2x + 3Cosec^2 3x}{(Cosec\,2x + Cot\,3x)^2}$$

$(11)\ y = (3 - 2Cot\,x)^3$

Answer

$$\frac{dy}{dx} = 3(3 - 2Cot\,x)^2\left[2Cosec^2 x\right] = 6Cosec^2 x(3 - 2Cot\,x)^2$$

$(12)\ y = \sqrt{1 + Cosec\,x}$

Answer

$$\frac{dy}{dx} = \frac{-Cosec\,x\,Cot\,x}{2\sqrt{1 + Cosec\,x}}$$

$(13)\ y = Cosec^3(2x + \pi)$

Answer

$$Let\ y = \left[Cosec\,(2x + \pi)\right]^3 \Rightarrow \therefore\ \frac{dy}{dx} = 3\left[Cosec\,(2x + \pi)\right]^2\times\left[-2Cosec(2x + \pi)Cot(2x + \pi)\right]$$

$$\frac{dy}{dx} = -6\,Cosec(2x + \pi)Cot(2x + \pi)\left[Cosec\,(2x + \pi)\right]^2$$

$(14)\ y = Cot^2(Sin\,\theta)$

Answer

$$\frac{dy}{dx} = 2\left[Cot(Sin\,\theta)\right]\times\left[-Cosec^2(Sin\,\theta)\times Cos\,\theta\right] = -2Cos\,\theta\ Cot(Sin\,\theta)Cosec^2(Sin\,\theta)$$

$(15)\ y = Cos^2(hx) + Sec^2(mx)$

Answer

$$Let\ y = [Cos(hx)]^2 + [Sec(mx)]^2$$

$$\therefore \frac{dy}{dx} = 2[Cos(hx)] \times [-h\ Sin(hx)] + 2[Sec(mx)] \times [m\ Sec(mx) Tan(mx)]$$

$$\therefore \frac{dy}{dx} = -2h\ Sin(hx) Cos(hx) + 2m\ Sec^2(mx)\ Tan(mx)$$

$$\therefore \frac{dy}{dx} = -\ h\ Sin\ 2(hx) + 2m\ Sec^2(mx)\ Tan(mx)$$

Example (16)

Find the first derivative of the function $f(x) = \frac{Sec\,x}{1 + Tan\,x}$, then determine the values of x when $f'(x) = 0$

Answer

$$f'(x) = \frac{(1+Tan\,x)(Sec\,x\,Tan\,x) - (Sec\,x)(Sec^2 x)}{(1+Tan\,x)^2} = \frac{Sec\,x\,Tan\,x + Sec\,x\,Tan^2 x - Sec^3 x}{(1+Tan\,x)^2}$$

$$f'(x) = \frac{Sec\,x[Tan\,x + Tan^2 x - Sec^2 x]}{(1+Tan\,x)^2} = \frac{Sec\,x[Tan\,x - 1]}{(1+Tan\,x)^2}$$

Notes

$\because 1 + Tan^2 x = Sec^2 x$

$\therefore Tan^2 x - Sec^2 x = -1$

$$When\ f'(x) = 0 \Rightarrow \therefore \frac{Sec\,x[Tan\,x - 1]}{(1+Tan\,x)^2} = 0$$

$\therefore Sec\,x[Tan\,x - 1] = 0 \Rightarrow$ so, either $Sec\,x = 0$ (refused)

Or $Tan\,x - 1 = 0 \Rightarrow \therefore Tan\,x = 1 \Rightarrow \therefore x = Tan^{-1} 1 = 45^o$ or 225^o or 405^o or ....

$$Then\ x = \left\{ \frac{\pi}{4} + n\pi \quad for\ every\ n \in Z \right\}$$

Example (17)

Find the first derivative of the function $f(x) = \frac{1 - Cosec\,x}{1 + Cosec\,x}$.

Answer

....

....

....

....

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$$Ans: \frac{2Cosec\,x\,Cot\,x}{(1 + Cosec\,x)^2}$$